Descriptive Complexity of Deterministic Polylogarithmic Time

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Descriptive Complexity

- Offers a machine independent description of complexity classes:
  - Time/Space used by a machine to decide a problem
    $\Rightarrow$ richness of the logical language needed to describe the problem.
- Complexity classes can/could be then separated by separating logics.
- Many characterisations are known:
  - Fagin's Theorem 1973: Existential second-order logic characterises NP.

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$$\text{"A graph is three colourable"} = \exists R \exists B \exists G \big(\text{"each node is labeled by exactly one colour"} \wedge \text{"adjacent nodes are always coloured with distinct colours"}\big)$$

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- Many characterisations are known:
  - Fagin's Theorem 1973: Existential second-order logic characterises NP.
  - $\mathrm{ESO}^{\mathrm{polylog}}$ characterises $\mathrm{NPolylogTime}$.
  - Second-order logic characterises the polynomial time hierarchy.
  - Least fixed point logic LFP characterises P on ordered structures.
  - ...
  - Major open problem: Does there exist a logic for P?

Sublinear Complexity Classes and Random Access Machines

- In sublinear time the whole the input cannot be read.
  - Turing machines with sequental access to the input does not suffice.
  - Instead random access model is used (cf. random access memory RAM)
- Random access machine model:

1	0	0	1	1	0	B	$\cdots$	**Input Tape (read only)**
1	0	B	B	B	B	B	$\cdots$	**Address Tape**
0	1	1	0	1	B	B	$\cdots$	**k Work Tapes**

  - Finite control of the machine as for TM.
- PolylogTIME $= \bigcup_{k \in \mathbb{N}} \mathrm{DTIME}[\log^k n]$

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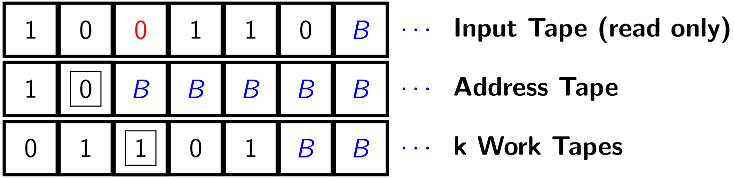

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Example computation in deterministic polylogarithmic time

- Calculate the length n of the input.

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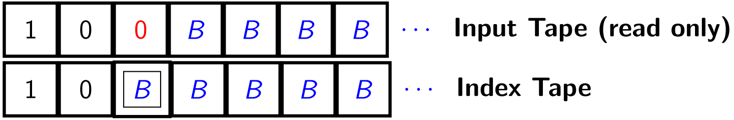

  - The index tape has $n - 1$ as binary.
- Any polynomial time numerical property of n (in binary) can be computed.

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Structures as inputs to the Turing machine

- Finite ordered structures with domain $\{0, \ldots, n\}$ and finite vocabularies.
- Structures are encoded as strings as usual in descriptive complexity.
- Relation $R^{\mathbf{A}}$ of arity k is encoded as a binary string of length $|A|^k$, where 1 in a given position indicates that the corresponding tuple is in the relation.
- Constant number $c^{\mathbf{A}}$ is encoded as a binary string of length $\lceil \log n \rceil$.
- k-ary functions are viewed as $\lceil \log n \rceil$-many k-ary relations, where the i-th relation indicates whether the i-th bit is 1.
- $\mathrm{DTIME}[\log^k \hat{n}] = \mathrm{DTIME}[\log^k n]$, where $\hat{n}$ is the length of the encoding and n the domain size.

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Index Logic

- Two sorted structures:
  - Domain of the structure: $\{0, \ldots, n\}$, for some n.
  - Built-in order predicate $\leq$ for the domain.
  - Functions, constants, relations and first-order variables range over the domain.
  - Numerical domain: $\{0, \ldots, \lceil \log n \rceil - 1\}$.
  - Built-in order predicate $\leq$ for the numerical domain.
  - First-order and second-order variables ranging over the numerical domain.
- Vars $x, y, \ldots$ range over the domain, and $\nu, \mu, \ldots$ over the numerical one.
- Idea: Full fixed point logic over the numerical sort, and restricted quantification over the actual domain.

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Fixpoints

Let $F\colon \mathcal{P}(B) \to \mathcal{P}(B)$ be a function.

- X is a fixed point of F, if $F(X) = X$.
- X is the least fixed point, if additionally $X \subseteq Y$ for all other fixed points Y.

For monotonic functions, the least fixed $\mathrm{lfp}(F)$ point always exists.
It can be calculated as the limit of the process:

$$F^0 = \emptyset, \quad F^{m+1} = F(F^m)$$

For non-monotonic functions, we may take the inflationary fixed point $\mathrm{ifp}(F)$.
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Fixed point logics

- Let $\varphi(X, \bar{x})$ be a formula with a free k-ary relation variable, and $\bar{x}$ a k-tuple of variables.
- On a model $\mathbf{A}, s$, the formula $\varphi(X, \bar{x})$ defines a function $F^{\mathbf{A},s}_{\varphi,X,\bar{x}} \colon \mathcal{P}(A^k) \to \mathcal{P}(A^k)$:

$$F^{\mathbf{A},s}_{\varphi,X,\bar{x}}(B) := \{\bar{a} \mid \mathbf{A}, s(X \mapsto B, \bar{x} \mapsto \bar{a}) \models \varphi\}.$$

- We may take the least fixed point or inflationary fixed point of $F^{\mathbf{A},s}_{\varphi,X,\bar{x}}$.

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Index Logic – Syntax

- Ordinary terms: $t ::= x \mid c \mid f(t, \ldots, t)$.
- Numerical terms: Only numerical variables μ, etc.

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Index Logic – Syntax

- Atomic formulae:

$$\varphi ::= t = t' \mid t \leq t' \mid \mu = \mu' \mid \mu \leq \mu' \mid R(t_1, \ldots, t_n) \mid X(\mu_1, \ldots, \mu_k) \mid$$

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- More atomic formulae

$$t = \mathit{index}\{\mu : \varphi(\mu)\} \mid [\mathrm{LFP}_{\bar{\mu}, X}\varphi]\bar{\nu} \mid$$

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- More atomic formulae

$$t = index\{\mu : \varphi(\mu)\} \mid [\mathrm{LFP}_{\bar{\mu}, X}\varphi]\bar{\nu} \mid$$

- Complex formulae

$$\varphi \land \varphi \mid \neg\varphi \mid \exists\mu\varphi \mid \exists x\big(x = index\{\mu : \alpha(\mu)\} \land \varphi\big)$$

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Index Logic – Semantics

$\mathbf{A}, s \models t_1 = t_2$ iff $s(t_1) = s(t_2)$

$\mathbf{A}, s \models t_1 \leq t_2$ iff $s(t_1) \leq s(t_2)$

$\mathbf{A}, s \models R(t_1, \ldots, t_k)$ iff $(s(t_1), \ldots, s(t_k)) \in R^{\mathbf{A}}$

$\mathbf{A}, s \models X(\mu_1, \ldots, \mu_k)$ iff $(s(\mu_1), \ldots, s(\mu_k)) \in s(X)$

$\mathbf{A}, s \models \neg\varphi$ iff $\mathbf{A}, s \not\models \varphi$

$\mathbf{A}, s \models \varphi \wedge \psi$ iff $\mathbf{A}, s \models \varphi$ and $\mathbf{A}, s \models \psi$

$\mathbf{A}, s \models \varphi \vee \psi$ iff $\mathbf{A}, s \models \varphi$ or $\mathbf{A}, s \models \psi$

$\mathbf{A}, s \models \exists\mu\,\varphi$ iff $\mathbf{A}, s(\mu \mapsto i) \models \varphi$, for some $i \leq \lceil \log|A| \rceil$

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Index Logic – Semantics

$\mathbf{A}, s \models t = index\{\mu : \varphi(\mu)\}$ iff

$s(t)$ in binary is $\bar{b}$, where the ith bit is 1 iff $\mathbf{A}, s(\mu \mapsto i) \models \varphi$

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Index Logic – Semantics

$\mathbf{A}, s \models \exists x(x = \mathit{index}\{\mu : \alpha(\mu)\} \land \varphi)$ iff

$\mathbf{A}, s(x \mapsto i) \models x = \mathit{index}\{\mu : \alpha(\mu)\} \land \varphi$, for some $i \in A$.

Index Logic – Semantics

$$\mathbf{A}, s \models [\mathrm{LFP}_{\bar{\mu},X}\varphi]\bar{\nu} \text{ iff } s(\bar{\nu}) \in \mathrm{lfp}(F^{\mathbf{A}}_{\varphi,\bar{\mu},X}).$$

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Theorem

Over ordered structures, index logic captures PolylogTime.

Theorem

Let c and d be constant symbols in a vocabulary σ. There does not exist an index logic formula φ that does not use the order predicate $\leq$ on ordinary terms and that is equivalent with the formula $c \leq d$.

Theorem

Let σ be a vocabulary without constant or function symbols. For every sentence φ of index logic of vocabulary σ there exists an equivalent sentence φ' that does not use the order predicate on ordinary terms.

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Proposition

Checking emptiness of a unary relation P^{A} is not computable in $\mathrm{PolylogTime}$*.*
Hence $\exists x P(x)$ is not expressible in index logic.

Proof.

- Let M be a TM that decides in $\mathrm{PolylogTime}$ whether $P^{\mathbf{A}}$ is empty.
  Let f be a polylogarithmic function that bounds the running time of M.
- Let $\mathbf{A}_{\emptyset}$ be the $\{P\}$-structure with domain $\{0, \dots, n-1\}$, where $P^{\mathbf{A}} = \emptyset$.
  The encoding of $\mathbf{A}_{\emptyset}$ to the Turing machine M is the sequence $s := \underbrace{0 \dots 0}_{n \text{ times}}$.
- The running time of M with input s is strictly less than n.
  Let i be an index of s that was not read in the computation $M(s)$.
- Define $s' := \underbrace{0 \dots 0}_{i \text{ times}} 1 \underbrace{0 \dots 0}_{n-i-1 \text{ times}}$.
- The output of the computations $M(s)$ and $M(s')$ are identical.

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Direct Access Turing Machines

- A novel variant on RAM that accesses the structure directly.
- For each k-ary relation

0	1	1	0	1	B	B	$\cdots$	**k Address Tapes**

- For each k-ary function

0	1	1	0	1	B	B	$\cdots$	**k Address Tapes**
0	1	1	0	1	B	B	$\cdots$	**1 Function Value Tape (Read Only)**

- Additionally

0	1	1	0	1	B	B	$\cdots$	**1 Extra Read Only Tape (stores $\|A\|$)**
0	1	1	0	1	B	B	$\cdots$	**k Work Tapes**

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- Order-invariant properties are properties of ordered models that remain unaffected if the ordering is redefined.
- Which order-invariant properties are computable in PolylogTime?
- E.g., any polynomial-time numerical property of the size of the domain is clearly computable. For example even cardinality is computable.
- The binary representation of a constant can be computed. However the number depends on the order.

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