# Characterizing Frame Definability in Team Semantics via The Universal Modality

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# PART I

# Definability

Which properties of graphs can be described with a given logic $\mathcal{L}$.

Example first-order logic on graphs $G = (V, E)$:

- Single formula: $\exists x \exists y \neg x = y$ defines the class $\{(V, E) \mid |V| \geq 2\}$.
- Set of formulae:

$$\{\exists x_1 \dots x_n \bigwedge_{i \neq j \leq n} \neg x_i = x_j \mid n \in \mathbb{N}\}$$

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# Modal logic

Set $\Phi$ of atomic propositions. The formulae of $\mathcal{ML}(\Phi)$ are generated by:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi.$$

Semantics via pointed Kripke structures $(W, R, V), w$. Nonempty set $W$, binary relation $R \subseteq W^2$, valuation $V : \Phi \to \mathcal{P}(W)$, point $w \in W$.

E.g.,

- $K, w \models p$ iff $w \in V(p)$,
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# Validity in models and frames

- Pointed model $(K, w)$: $(W, R, V), w$.
- Model $(K)$: $(W, R, V)$.
- Frame $(F)$: $(W, R)$.

We write:

- $(W, R, V) \models \varphi$ iff $(W, R, V), w \models \varphi$ holds for every $w \in W$.
- $(W, R) \models \varphi$ iff $(W, R, V) \models \varphi$ holds for every valuation $V$.

Every (set of) $\mathcal{ML}$-formula defines the class of frames in which it is valid.

- $Fr(\varphi) := \{(W, R) \mid (W, R) \models \varphi\}$.
- $Fr(\Gamma) := \{(W, R) \mid \forall \varphi \in \Gamma : (W, R) \models \varphi\}$.

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# Frame definability

Which classes of Kripke frames are definable by a (set of) modal formulae.

Which elementary classes are definable by a (set of) modal formulae.

Examples:

| Formula | Property of $R$ | |
|---|---|---|
| $\Box p \rightarrow p$ | Reflexive | $\forall w\,(wRw)$ |
| $p \rightarrow \Box\Diamond p$ | Symmetric | $\forall w, v\,(wRv \rightarrow vRw)$ |
| $\Box p \rightarrow \Box\Box p$ | Transitive | $\forall w, v, u\,((wRv \wedge vRu) \rightarrow wRu)$ |
| $\Diamond p \rightarrow \Box\Diamond p$ | Euclidean | $\forall w, v, u\,((wRv \wedge wRu) \rightarrow vRu)$ |
| $\Box p \rightarrow \Diamond p$ | Serial | $\forall w \exists v\,(wRv)$ |

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# Goldblatt-Thomason Theorem (1975)

Set $\Phi$ of atomic propositions. The formulae of $\mathcal{ML}(\Phi)$ are generated by:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi.$$

**Theorem**

*An elementary frame class is $\mathcal{ML}$-definable iff*

- *it is closed under taking*
  - *bounded morphic images*
  - *generated subframes*
  - *disjoint unions*
- *and its complement is closed under taking*
  - *ultrafilter extensions.*

# Goldblatt-Thomason Theorem (Goranko, Passy 1992)

The formulae of $\mathcal{ML}(\boxed{u})$ are generated by:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi \mid \boxed{u}\,\varphi.$$

$$K, w \models \boxed{u}\,\varphi \quad \leftrightarrow \quad \forall v \in W : K, v \models \varphi.$$

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# What do we study?

Frame definability of the fragment $\mathcal{ML}(\boxed{\text{u}}^{+})$ of $\mathcal{ML}(\boxed{\text{u}})$:

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Box\varphi \mid \Diamond\varphi \mid \boxed{\text{u}}\,\varphi.$$

Frame definability of particular team based modal logics:

- Modal dependence logic $\mathcal{MDL}$.
- Extended modal dependence logic $\mathcal{EMDL}$.
- Modal logic with intuitionistic disjunction $\mathcal{ML}(\varovee)$.

# What do we show?

- We give a variant of the Goldblatt-Thomason theorem for $\mathcal{ML}(\boxed{\mathsf{u}}^+)$.
- We show that with respect to frame definability:

$$\mathcal{ML} < \mathcal{MDL} = \mathcal{EMDL} = \mathcal{ML}(\varovee) = \mathcal{ML}(\boxed{\mathsf{u}}^+) < \mathcal{ML}(\boxed{\mathsf{u}}).$$

# Frame definability in $\mathcal{ML}(\boxed{u}^+)$

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Every $\mathcal{ML}$-definable class is $\mathcal{ML}(\boxed{u}^+)$-definable, but not vice versa.

$\mathcal{ML}(\boxed{u}^+)$ is not closed under disjoint unions (e.g., $\boxed{u}\, p \lor \boxed{u}\, \neg p$).

Therefore $\mathcal{ML} <_F \mathcal{ML}(\boxed{u}^+)$.

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# Goldblatt-Thomason Theorem for $\mathcal{ML}(⊔^+)$

**Theorem (Does this suffice?)**

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NO! Something more is needed.

# Goldblatt-Thomason Theorem for $\mathcal{ML}(⧈^+)$

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NO! Something more is needed.

# Reflection of Finitely Generated Subframes

A frame class $\mathbb{F}$ reflects finitely generated subframes if:
whenever every finitely generated subframe of $\mathfrak{F}$ is in $\mathbb{F}$, then $\mathfrak{F}$ is also in $\mathbb{F}$.

**Theorem**

*Every $\mathcal{ML}(\boxed{u}^+)$-definable frame class $\mathbb{F}$ reflects finitely generated subframes.*

# Goldblatt-Thomason theorem for $\mathcal{ML}(\boxed{u}^+)$

**Theorem**

An elementary frame class $\mathbb{F}$ is $\mathcal{ML}(\boxed{u}^+)$-definable iff $\mathbb{F}$ is closed under taking

- bounded morphic images & generated subframes

and it reflects

- ultrafilter extensions & finitely generated subframes.

$\because$ By van Benthem (1993)'s model theoretic argument.

# PART II

# Team Semantics: Motivation and history

Logical modelling of uncertainty, imperfect information and functional dependence in the framework of modal logic.

The ideas are transfered from first-order dependence logic (and independence-friendly logic) to modal logic.

Historical development:

- Branching quantifiers by Henkin 1959.
- Independence-friendly logic by Hintikka and Sandu 1989.
- Compositional semantics for independence-friendly logic by Hodges 1997. (Origin of team semantics.)
- IF modal logic by Tulenheimo 2003.
- Dependence logic by Väänänen 2007.
- Modal dependence logic by Väänänen 2008.

# Syntax for modal logic in negation normal form

## Definition

Let $\Phi$ be a set of atomic propositions. The set of formulae for $\mathcal{ML}(\Phi)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \Phi$.

Negations may occur only in front of atomic formulae.

# Team semantics?

1. In this context a team is a set of possible worlds, i.e., if $K = (W, R, V)$ is a Kripke model then $T \subseteq W$ is a team of $K$.
2. The standard semantics for modal logic is given with respect to pointed models $K, w$. In team semantics the semantics is given for models and teams, i.e., with respect to pairs $K, T$, where $T$ is a team of $K$.
3. Some possible interpretations for $K, w$ and $K, T$:
   - (a) $K, w \models \varphi$: The actual world is $w$ and $\varphi$ is true in $w$.
   - (b) $K, T \models \varphi$: The actual world is in $T$, but we do not know which one it is. The formula $\varphi$ is true in the actual world.
   - (c) $K, T \models \varphi$: We consider sets of points as primitive. The formula $\varphi$ describes properties of collections of points.

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# Team semantics for modal logic

## Definition

Kripke/Team semantics for $\mathcal{ML}$ is defined as follows. Remember that $K = (W, R, V)$ is a normal Kripke model and $T \subseteq W$.

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\begin{aligned}
K, w \models p \quad &\Leftrightarrow \quad w \in V(p). \\
K, w \models \neg p \quad &\Leftrightarrow \quad w \notin V(p). \\
K, w \models \varphi \wedge \psi \quad &\Leftrightarrow \quad K, w \models \varphi \text{ and } K, w \models \psi. \\
K, w \models \varphi \vee \psi \quad &\Leftrightarrow \quad K, w \models \varphi \text{ or } K, w \models \psi. \\
K, w \models \Box\varphi \quad &\Leftrightarrow \quad K, w' \models \varphi \text{ for every } w' \text{ s.t. } wRw'. \\
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# Team semantics for modal logic

## Definition

Kripke/Team semantics for $\mathcal{ML}$ is defined as follows. Remember that $K = (W, R, V)$ is a normal Kripke model and $T \subseteq W$.

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K, T \models p & \Leftrightarrow & T \subseteq V(p). \\
K, T \models \neg p & \Leftrightarrow & T \cap V(p) = \emptyset. \\
K, T \models \varphi \wedge \psi & \Leftrightarrow & K, T \models \varphi \text{ and } K, T \models \psi. \\
K, T \models \varphi \vee \psi & \Leftrightarrow & K, T_1 \models \varphi \text{ and } K, T_2 \models \psi \text{ for some } T_1 \cup T_2 = T. \\
K, T \models \Box\varphi & \Leftrightarrow & K, T' \models \varphi \text{ for } T' := \{w' \mid w \in T,\ wRw'\}. \\
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$$\forall w \in T \, \exists w' \in T' : wRw' \text{ and } \forall w' \in T' \, \exists w \in T : wRw'.$$

Note that $K, \emptyset \models \varphi$ for every formula $\varphi$.

# Team semantics vs. Kripke semantics

## Theorem (Flatness property of ML)

*Let $K$ be a Kripke model, $T$ a team of $K$ and $\varphi$ a $\mathcal{ML}$-formula. Then*

$$K, T \models \varphi \quad \Leftrightarrow \quad K, w \models \varphi \textit{ for all } w \in T,$$

*in particular*

$$K, \{w\} \models \varphi \quad \Leftrightarrow \quad K, w \models \varphi.$$

Note that it also follows that every $\mathcal{ML}$-formula is *downwards closed*:

If $K, T \models \varphi$ and $S \subseteq T$, then $K, S \models \varphi$.

# Modal dependence logic

Introduced by Väänänen 2008, the syntax modal dependence logic $\mathcal{MDL}$ extends the syntax of modal logic by the clause

$$\mathrm{dep}(p_1, \ldots, p_n, q)\,,$$

where $p_1, \ldots, p_n, q$ are proposition symbols.

The intended meaning of the atomic formula

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$$K, T \models \mathrm{dep}(p_1, \ldots, p_n, q)$$

if and only if $\forall w_1, w_2 \in T$:

$$\bigwedge_{i \leq n} (w_1 \in V(p_i) \Leftrightarrow w_2 \in V(p_i)) \Rightarrow (w_1 \in V(q) \Leftrightarrow w_2 \in V(q)).$$

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# Intuitionistic disjunction

$\mathcal{ML}(\varovee)$: add a different version of disjunction $\varovee$ to modal logic with the semantics:

- $K, T \models \varphi \varovee \psi \iff K, T \models \varphi$ or $K, T \models \psi$.

Dependence atoms are definable in $\mathcal{ML}(\varovee)$ (Väänänen 09):

$$K, T \models \mathrm{dep}(p_1, \ldots, p_n, q) \iff K, T \models \bigvee_{s \in F}(\theta_s \wedge (q \varovee \neg q)),$$

where $F$ is the set of all $\{p_1, \ldots, p_n\}$-assignments, and $\theta_s$ is the formula $\bigwedge_{i \leq n} p_i^{s(p_i)}$, where $p_i^{\perp} = \neg p_i$ and $p_i^{\top} = p_i$.

# Expressive Power

**Theorem (Ebbing, Hella, Meier, Müller, V., Vollmer 13)**

$\mathcal{MDL} < \mathcal{EMDL} \leq \mathcal{ML}(\varovee)$.

**Theorem (Hella, Luosto, Sano, V. 14)**

$\mathcal{ML}(\varovee) \leq \mathcal{EMDL}$. *Consequently,* $\mathcal{EMDL} \equiv \mathcal{ML}(\varovee)$.

**Theorem (Gabbay, van Benthem)**

*A class $\mathcal{C}$ of pointed Kripke models is definable in $\mathcal{ML}$ if and only if $\mathcal{C}$ is closed under $k$-bisimulation for some $k \in \mathbb{N}$.*

**Theorem (Hella, Luosto, Sano, V. 14)**

*A nonempty class $\mathcal{C}$ is definable in $\mathcal{ML}(\varovee)$ if and only if $\mathcal{C}$ is downward closed and there exists $k \in \mathbb{N}$ such that $\mathcal{C}$ is closed under team $k$-bisimulation.*

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# Frame definability in team semantics

Def. $K \models \varphi$ iff $\forall T \subseteq W : K, T \models \varphi$ (iff $K, W \models \varphi$)

It is easy to show that $\mathcal{MDL} =_F \mathcal{EMDL}$.

**Proof**

Let $\varphi$ be the dependence atom $\mathrm{dep}(\psi_1, \ldots, \psi_n)$, let $k$ be the modal depth of $\varphi$, and let $p_1, \ldots, p_n$ be distinct fresh proposition symbols. Define

$$\varphi^* := \Big( \bigwedge_{0 \leq i \leq k} \Box^i \bigwedge_{1 \leq j \leq n} (p_j \leftrightarrow \psi_j)\Big) \to \mathrm{dep}(p_1, \ldots, p_n).$$

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# Normal Form for $\mathcal{ML}(\boxed{\mathrm{u}}^+)$

Similar to the normal form for $\mathcal{ML}(\boxed{\mathrm{u}})$ by Goranko and Passy 1992.

A formula $\varphi$ is a closed disjunctive $\boxed{\mathrm{u}}$-clause if

$\varphi$ is of the form $\bigvee_{i \in I} \boxed{\mathrm{u}}\, \psi_i$ $(\psi_i \in \mathcal{ML})$.

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**Theorem**

*Each formula of $\mathcal{ML}(\boxed{\mathrm{u}}^+)$ is equivalent to a formula in conjunctive $\boxed{\mathrm{u}}$-form.*

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With respect to frame definability $\mathcal{ML}(\boxed{\mathrm{u}}^+)$ and $\bigvee \boxed{\mathrm{u}}\, \mathcal{ML}$ coincide.

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# Results

**Theorem**

An elementary frame class $\mathbb{F}$ is $\mathcal{L}$-definable
($\mathcal{L} \in \{\mathcal{ML}(\varovee), \mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\boxed{u}^+)\}$) iff
$\mathbb{F}$ is closed under taking

- bounded morphic images & generated subframes

and it reflects

- ultrafilter extensions & finitely generated subframes.

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With respect to frame definability:
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# References


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# Bounded morphism and Ultrafilter Extension

$f : (W, R) \to (W', R')$ is a bounded morphism if:

- (Forth) $wRv$ implies $f(w)R'f(v)$
- (Back) $f(w)R'b$ implies: $f(v) = b$ and $wRv$ for some $v$

$(\mathrm{Uf}(W), R^{\mathrm{ue}})$ is the ultrafilter extension of $(W, R)$ where:

- $\mathrm{Uf}(W)$ is the set of all ultrafilters $\mathcal{U} \subseteq \mathcal{P}(W)$.
- $\mathcal{U} R^{\mathrm{ue}} \mathcal{U}'$ iff $Y \in \mathcal{U}'$ implies $R^{-1}[Y] \in \mathcal{U}$ for all $Y \subseteq W$.