Characterizing Relative Frame Definability in Team Semantics via The Universal Modality

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PART I

Definability

Which properties of graphs can be described with a given logic $\mathcal{L}$.

Example first-order logic on graphs $G = (V, E)$:

- Single formula: $\exists x \exists y \, \neg x = y$ defines the class $\{(V, E) \mid |V| \geq 2\}$.
- Set of formulae:

$$\{\exists x_1 \dots x_n \bigwedge_{i \neq j \leq n} \neg x_i = x_j \mid n \in \mathbb{N}\}$$

  defines the class of infinite graphs.

Relative Definability

Which properties of graphs can be described with a given logic $\mathcal{L}$ if restricted to some class of graphs $\mathcal{C}$.

Example monadic second-order logic on finite ordered graphs $G = (V, E)$:

- Single formula:
  $$\exists X \Big(\exists xy \big(\mathrm{min}(x) \wedge X(x) \wedge \mathrm{max}(y) \wedge X(y)\big) \wedge \forall xy \big(\mathrm{succ}(x, y) \rightarrow (X(x) \leftrightarrow \neg X(y))\big) \Big)$$
  defines the class $\{(V, E) \mid |V| \text{ is odd}\}$ relative to finite ordered graphs.

Modal logic

Set Φ of atomic propositions. The formulae of $\mathcal{ML}(\Phi)$ are generated by:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi.$$

Semantics via pointed Kripke structures $(W, R, V), w$. Nonempty set W, binary relation $R \subseteq W^2$, valuation $V : \Phi \to \mathcal{P}(W)$, point $w \in W$.

E.g.,

- $K, w \models p$ iff $w \in V(p)$,
- $K, w \models \Diamond\varphi$ iff $K, v \models \varphi$ for some v s.t. wRv.

Validity in models and frames

- Pointed model (K, w): $(W, R, V), w$.
- Model (K): (W, R, V).
- Frame (F): (W, R).

We write:

- $(W, R, V) \models \varphi$ iff $(W, R, V), w \models \varphi$ holds for every $w \in W$.
- $(W, R) \models \varphi$ iff $(W, R, V) \models \varphi$ holds for every valuation V.

Every (set of) $\mathcal{ML}$-formula defines the class of frames in which it is valid.

- $Fr(\varphi) := \{(W, R) \mid (W, R) \models \varphi\}$.
- $Fr(\Gamma) := \{(W, R) \mid \forall \varphi \in \Gamma : (W, R) \models \varphi\}$.

Frame definability

Which classes of frames are definable by a (set of) modal formulae.

Which classes are definable by a (set of) modal formulae within the class $\mathcal{F}_{\text{fintra}}$ of finite transitive frames.

Examples:

Formula	Property of R	
$\Box p \to p$	Reflexive	$\forall w\,(wRw)$
$p \to \Box\Diamond p$	Symmetric	$\forall wv\,(wRv \to vRw)$
$\Box p \to \Box\Box p$	Transitive	$\forall wvu\,((wRv \land vRu) \to wRu)$
$\Diamond p \to \Box\Diamond p$	Euclidean	$\forall wvu\,((wRv \land wRu) \to vRu)$
$\Box p \to \Diamond p$	Serial	$\forall w \exists v\,(wRv)$
$\Box(\Box p \to p) \to \Box p$	Irreflexive w.r.t $\mathcal{F}_{\text{fintra}}$	$\forall wv\,\neg(wRv)$

Goldblatt-Thomason Theorem (1975)

Set Φ of atomic propositions. The formulae of $\mathcal{ML}(\Phi)$ are generated by:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi.$$

Theorem

An elementary frame class is $\mathcal{ML}$-definable iff

- *it is closed under taking*
  - *bounded morphic images*
  - *generated subframes*
  - *disjoint unions*
- *and its complement is closed under taking*
  - *ultrafilter extensions.*

Goldblatt-Thomason Theorem (Goranko, Passy 1992)

The formulae of $\mathcal{ML}(\boxed{u})$ are generated by:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi \mid \boxed{u}\,\varphi.$$

$$K, w \models \boxed{u}\,\varphi \quad \leftrightarrow \quad \forall v \in W : K, v \models \varphi.$$

Theorem

An elementary frame class is $\mathcal{ML}(\boxed{u})$-definable iff

- *it is closed under taking*
  - *bounded morphic images*
- *and its complement is closed under taking*
  - *ultrafilter extensions.*

Goldblatt-Thomason Theorem (Sano, V. 2015)

The formulae of $\mathcal{ML}(\boxed{u}^+)$ are generated by:

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Box\varphi \mid \Diamond\varphi \mid \boxed{u}\,\varphi.$$

Theorem

An elementary frame class is $\mathcal{ML}(\boxed{u}^+)$-definable iff

- *it is closed under taking*
  - *bounded morphic images*
  - *generated subframes*
- *and it reflects*
  - *ultrafilter extensions,*
  - *finitely generated subframes.*

Goldblatt-Thomason Theorem in the Finite

Theorem (van Benthem 1988)

A class of finite transitive frames is $\mathcal{ML}$-definable within the class $\mathbb{F}_{\mathrm{fintra}}$ of all finite transitive frames if and only if it is closed under taking

- *bounded morphic images,*
- *generated subframes,*
- *disjoint unions.*

Theorem (Gargov, Goranko 1993)

A class of finite frames is $\mathcal{ML}(\boxed{u})$-definable within the class $\mathbb{F}_{\mathrm{fin}}$ of all finite frames if and only if it is closed under taking bounded morphic images.

What do we study?

- Frame definability in $\mathcal{ML}(\boxed{u}^+)$ within finite transitive frames.
- Frame (model) definability of particular team based modal logics:
  - Modal dependence logics $\mathcal{MDL}$ and $\mathcal{EMDL}$.
  - Modal inclusion logics $\mathcal{MINC}$ and $\mathcal{EMINC}$.
  - Modal team logic $\mathcal{MTL}$.
- Note: Frame (model) definability of $\mathcal{ML}(\boxed{u}^+)$ coincides with that of $\mathcal{EMDL}$ (Sano, V. 2015).

What do we show?

- Variant of the Goldblatt-Thomason theorem for $\mathcal{ML}(\boxdot^+)$ within $\mathbb{F}_{\text{fintra}}$.
- We show the following trichotomy with respect to model definability:

$$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} < \mathcal{MDL} < \{\mathcal{EMDL}, \mathcal{ML}(\boxdot^+), \mathcal{MTL}\}$$

- We show the following dichotomy with respect to frame definability:

$$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} < \{\mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\boxdot^+), \mathcal{MTL}\}.$$

The expressive powers of all of the logics above differ.

Frame definability in $\mathcal{ML}(\boxed{\text{u}}^+)$ within finite transitive frames

Theorem

A class of finite transitive frames is $\mathcal{ML}(\boxed{\text{u}}^+)$-definable within the class $\mathbb{F}_{\mathrm{fintra}}$ of all finite transitive frames if and only if it is closed under taking

- *bounded morphic images,*
- *generated subframes.*

The proof uses Jankov-Fine formulas $\varphi_{\mathfrak{F}}$ of the type $\bigvee_{w \in \mathrm{dom}(\mathfrak{F})} \boxed{\text{u}} \neg \varphi_{\mathfrak{F},w}$.

PART II

Team Semantics: Motivation and history

Logical modelling of uncertainty, imperfect information and functional, inclusion, etc., dependence in the framework of modal logic.

The ideas are transfered from first-order dependence logic (and independence-friendly logic) to modal logic.

Historical development:

- Branching quantifiers by Henkin 1959.
- Independence-friendly logic by Hintikka and Sandu 1989.
- Compositional semantics for independence-friendly logic by Hodges 1997. (Origin of team semantics.)
- IF modal logic by Tulenheimo 2003.
- Dependence logic by Väänänen 2007.
- Modal dependence logic by Väänänen 2008.

Syntax for modal logic in negation normal form

Definition

Let Φ be a set of atomic propositions. The set of formulae for $\mathcal{ML}(\Phi)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \Phi$.

Negations may occur only in front of atomic formulae.

Team semantics?

1. In this context a team is a set of possible worlds, i.e., if $K = (W, R, V)$ is a Kripke model then $T \subseteq W$ is a team of K.
2. The standard semantics for modal logic is given with respect to pointed models K, w. In team semantics the semantics is given for models and teams, i.e., with respect to pairs K, T, where T is a team of K.
3. Some possible interpretations for K, w and K, T:
   (a) $K, w \models \varphi$: The actual world is w and φ is true in w.
   (b) $K, T \models \varphi$: The actual world is in T, but we do not know which one it is. The formula φ is true in the actual world.
   (c) $K, T \models \varphi$: We consider sets of points as primitive. The formula φ describes properties of collections of points.

Team semantics for modal logic

Definition

Kripke/Team semantics for $\mathcal{ML}$ is defined as follows. Remember that $K = (W, R, V)$ is a normal Kripke model and $T \subseteq W$.

$$
\begin{aligned}
K, w \models p \quad &\Leftrightarrow \quad w \in V(p).\\
K, w \models \neg p \quad &\Leftrightarrow \quad w \notin V(p).\\
K, w \models \varphi \wedge \psi \quad &\Leftrightarrow \quad K, w \models \varphi \text{ and } K, w \models \psi.\\
K, w \models \varphi \vee \psi \quad &\Leftrightarrow \quad K, w \models \varphi \text{ or } K, w \models \psi.\\
K, w \models \Box\varphi \quad &\Leftrightarrow \quad K, w' \models \varphi \text{ for every } w' \text{ s.t. } wRw'.\\
K, w \models \Diamond\varphi \quad &\Leftrightarrow \quad K, w' \models \varphi \text{ for some } w' \text{ s.t. } wRw'.
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\begin{aligned}
K, T \models p \quad &\Leftrightarrow \quad T \subseteq V(p). \\
K, T \models \neg p \quad &\Leftrightarrow \quad T \cap V(p) = \emptyset. \\
K, T \models \varphi \wedge \psi \quad &\Leftrightarrow \quad K, T \models \varphi \text{ and } K, T \models \psi. \\
K, w \models \varphi \vee \psi \quad &\Leftrightarrow \quad K, w \models \varphi \text{ or } K, w \models \psi. \\
K, w \models \Box\varphi \quad &\Leftrightarrow \quad K, w' \models \varphi \text{ for every } w' \text{ s.t. } wRw'. \\
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K, T \models \varphi \vee \psi &\quad\Leftrightarrow\quad K, T_1 \models \varphi \text{ and } K, T_2 \models \psi \text{ for some } T_1 \cup T_2 = T.\\
K, w \models \Box\varphi &\quad\Leftrightarrow\quad K, w' \models \varphi \text{ for every } w' \text{ s.t. } wRw'.\\
K, w \models \Diamond\varphi &\quad\Leftrightarrow\quad K, w' \models \varphi \text{ for some } w' \text{ s.t. } wRw'.
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K, T \models \Box\varphi & \Leftrightarrow & K, T' \models \varphi \text{ for } T' := \{w' \mid w \in T, \ wRw'\}. \\
K, w \models \Diamond\varphi & \Leftrightarrow & K, w' \models \varphi \text{ for some } w' \text{ s.t. } wRw'.
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Team semantics for modal logic

Definition

Kripke/Team semantics for $\mathcal{ML}$ is defined as follows. Remember that $K = (W, R, V)$ is a normal Kripke model and $T \subseteq W$.

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K, T \models \Box\varphi \quad &\Leftrightarrow \quad K, T' \models \varphi \text{ for } T' := \{w' \mid w \in T,\ wRw'\}.\\
K, T \models \Diamond\varphi \quad &\Leftrightarrow \quad K, T' \models \varphi \text{ for some } T' \text{ s.t.}\\
&\qquad \forall w \in T\, \exists w' \in T' : wRw' \text{ and } \forall w' \in T'\, \exists w \in T : wRw'.
\end{aligned}
$$

Note that $K, \emptyset \models \varphi$ for every formula φ.

(Extended) Modal dependence logic

The syntax extended modal dependence logic $\mathcal{EMDL}$ extends the syntax of modal logic by the clause

$$\mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi)\,,$$

where $\varphi_1, \ldots, \varphi_n, \psi$ are $\mathcal{ML}$-formulae.

The intended meaning of the atomic formula

$$\mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi)$$

is that inside a team the truth values of the formulae $\varphi_1, \ldots, \varphi_n$ functionally determine the truth value of the formula ψ.

In $\mathcal{MDL}$ the formulae $\varphi_1, \ldots, \varphi_n, \psi$ above are proposition symbols.

Intuitionistic disjunction

$\mathcal{ML}(\varovee)$: add a different version of disjunction $\varovee$ to modal logic with the semantics:

- $K, T \models \varphi \varovee \psi \iff K, T \models \varphi$ or $K, T \models \psi$.

Dependence atoms are definable in $\mathcal{ML}(\varovee)$ (Väänänen 09):

Theorem (Hella, Luosto, Sano, V. 14)

With respect to expressive power $\mathcal{EMDL} \equiv \mathcal{ML}(\varovee)$.

(Extended) modal inclusion logic

The syntax extended modal inclusion logic $\mathcal{EMINC}$ extends the syntax of modal logic by the clause

$$\varphi ::= \varphi_1, \dots, \varphi_n \subseteq \psi_1, \dots, \psi_n,$$

where $\varphi_1, \psi_1, \dots, \varphi_n, \psi_n$ are $\mathcal{ML}$-formulae.

The meaning of the inclusion atom

$$\varphi_1, \dots, \varphi_n \subseteq \psi_1, \dots, \psi_n$$

is that the truth values that occur in a given team for the tuple $\varphi_1, \dots, \varphi_n$ occur also as truth values for the tuple $\psi_1, \dots \psi_n$.

In $\mathcal{MINC}$ the formulae $\varphi_1, \psi_1, \dots, \varphi_n, \psi_n$ above are proposition symbols.

Contradictory negation

$\mathcal{MTL}$: add a different version of negation $\sim$ to modal logic with the semantics:

- $K, T \models {\sim}\varphi \iff K, T \not\models \varphi$.

Theorem (Kontinen, Müller, Schnoor, Vollmer 2015)

A class of team pointed Kripke models if definable in $\mathcal{MTL}$ iff it is closed under team k-bisimulation for some finite k.

Frame definability in team semantics

$(W, R, V) \models \varphi$ iff $(W, R, V), T \models \varphi$ for all $T \subseteq W$.
$(W, R) \models \varphi$ iff $(W, R, V) \models \varphi$ for all valuations V.

Theorem (Sano, V. 2015)

With respect to frame definability:
$\mathcal{ML} < \{\mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\owedge), \mathcal{ML}(\boxdot^{+})\} < \mathcal{ML}(\boxdot)$.

Question

Where do $\mathcal{MINC}$, $\mathcal{EMINC}$, and $\mathcal{MTL}$ lie?

Theorem

With respect to frame definability:
$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} < \{\mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\owedge), \mathcal{ML}(\boxdot^{+}), \mathcal{MTL}\}$.

Hintikka formulae and types

Definition

Assume that Φ is a finite set of proposition symbols. Let $k \in \mathbb{N}$ and let (K, w) be a pointed Φ-model. The *k-th Hintikka formula* $\chi^k_{K,w}$ of (K, w) is defined recursively as follows:

- $\chi^0_{K,w} := \bigwedge\{p \mid p \in \Phi, w \in V(p)\} \wedge \bigwedge\{\neg p \mid p \in \Phi, w \notin V(p)\}$.
- $\chi^{k+1}_{K,w} := \chi^k_{K,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{K,v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{K,v}$.

Definition

Let K be a Kripke Φ-model and $\mathbb{C}$ a class of Kripke Φ-models. We define that

- $\mathrm{tp}^\Phi_k(K) := \{\chi^k_{K,w} \mid w \text{ is a point of } K\}$,
- $\mathrm{tp}^\Phi_k(K, T) := \{\chi^k_{K,w} \mid w \in T\}$,
- $\mathrm{tp}^\Phi_k(\mathbb{C}) := \{\mathrm{tp}^\Phi_k(K) \mid K \in \mathbb{C}\}$.

Model and frame definability of $\mathcal{MINC}$ and $\mathcal{ML}$ coincide

Lemma

Let Φ be a finite set of proposition symbols, $\varphi \in \mathcal{EMINC}(\Phi)$, and $k = \mathrm{md}(\varphi)$. Then $K \in \mathrm{Mod}(\varphi)$ iff $\mathrm{tp}_k^{\Phi}(K) \subseteq \bigcup\{\mathrm{tp}_k^{\Phi}(K') \mid K' \in \mathrm{Mod}(\varphi)\}$.

Theorem

A class $\mathbb{C}$ of Kripke models is definable by a single $\mathcal{EMINC}$-formula if and only if the class if definable by a single $\mathcal{ML}$-formula.

Let φ be an $\mathcal{EMINC}(\Phi)$-formula that defines $\mathbb{C}$. Let k denote the modal depth of φ. The $\mathcal{ML}(\Phi)$ formula

$$\varphi^* := \bigvee\{\chi_{K,w}^k \mid K \in \mathbb{C}, w \in K\}$$

defines $\mathbb{C}$.

Model and frame definability of $\mathcal{MTL}$ and $\mathcal{ML}(\varovee)$ coincide

Lemma

Let φ be and $\mathcal{MTL}$-formula and $k = \mathrm{md}(\varphi)$. Then

$$K \in \mathrm{Mod}(\varphi) \text{ iff } \mathrm{tp}_k^{\Phi}(K) \subseteq \Gamma \in \mathrm{tp}_k^{\Phi}(\mathrm{Mod}(\varphi)), \text{ for some } \Gamma.$$

Theorem

A class $\mathbb{C}$ of Kripke models is definable in $\mathcal{MTL}$ by a single formula if and only if it is definable in $\mathcal{ML}(\varovee)$ by a single formula.

Let φ be an $\mathcal{MTL}$-formula that defines $\mathbb{C}$. Let k denote the modal depth of φ. The $\mathcal{ML}(\varovee)$-formula

$$\varphi^* := \bigovee_{\Gamma \in \mathrm{tp}_k^{\Phi}(\mathbb{C})} \left(\bigvee \Gamma \right)$$

defines $\mathbb{C}$

Results

Thanks!

Theorem

A class of finite transitive frames is $\mathcal{ML}(\boxed{u}^+)$-definable within the class $\mathbb{F}_{\text{fintra}}$ of all finite transitive frames if and only if it is closed under taking

- *bounded morphic images,*
- *generated subframes.*

Theorem

The following trichotomy holds with respect to model definability:

$$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} < \mathcal{MDL} < \{\mathcal{EMDL}, \mathcal{ML}(\mathbin{\ovee}), \mathcal{ML}(\boxed{u}^+), \mathcal{MTL}\}$$

The following dichotomy holds with respect to frame definability:

$$\{\mathcal{ML}, \mathcal{MINC}, \mathcal{EMINC}\} < \{\mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\mathbin{\ovee}), \mathcal{ML}(\boxed{u}^+), \mathcal{MTL}\}.$$

Examples

- $\mathrm{dep}(p)$ defines the class of frames of cardinality 1.
- $\boxed{\mathrm{u}}\, p \vee \boxed{\mathrm{u}}\, \neg p$ defines the class of frames of cardinality 1.
- $p \subseteq \Diamond p$ defines the class $\{(W,R) \mid R = \{(w,w) \mid w \in W\}\}$.
- $\Box p \leftrightarrow p$ defines the class $\{(W,R) \mid R = \{(w,w) \mid w \in W\}\}$.

Bounded morphism and Ultrafilter Extension

$f : (W, R) \to (W', R')$ is a bounded morphism if:

- (Forth) wRv implies $f(w)R'f(v)$
- (Back) $f(w)R'b$ implies: $f(v) = b$ and wRv for some v

$(\mathrm{Uf}(W), R^{\mathrm{ue}})$ is the ultrafilter extension of (W, R) where:

- $\mathrm{Uf}(W)$ is the set of all ultrafilters $\mathcal{U} \subseteq \mathcal{P}(W)$.
- $\mathcal{U} R^{\mathrm{ue}} \mathcal{U}'$ iff $Y \in \mathcal{U}'$ implies $R^{-1}[Y] \in \mathcal{U}$ for all $Y \subseteq W$.