Probabilistic Team Semantics

Jonni Virtema

Hasselt University, Belgium
jonni.virtema@gmail.com

Joint work with Arnaud Durand (Université Paris Diderot), Miika Hannula (University of Helsinki), Juha Kontinen (University of Helsinki), and Arne Meier (Leibniz Universität Hannover)

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Teams as collections of measurements

- Multiteams (multisets of assignments) vs.

	x	y	z
s_1	a	a	b
s_2	a	a	b
s_3	b	c	c
s_4	a	b	c

	x	y	z	#
s_1	a	a	b	2
s_2	b	c	c	1
s_3	a	b	c	1

	x	y	z	prob.
s_1	a	a	b	$\frac{1}{2}$
s_2	b	c	c	$\frac{1}{4}$
s_3	a	b	c	$\frac{1}{4}$

Teams as collections of measurements

- Multiteams (multisets of assignments) vs. probabilistic teams (distributions over assignments)

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Distributions of data

Consider:

- A collection of data from some repetitive science experiment.
- Data obtained from a poll.
- Any collection of data, that involves meaningful duplicates of data.

One natural way to represent the data is to use multisets (sets with duplicates).

Claim:

Often the multiplicities themselves are not important; the distribution of data is.

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Distributions

Definition

A distribution is a mapping $f : A \to \mathbb{Q}_{[0,1]}$ from a set A of values to the closed interval $[0, 1]$ of rational numbers such that the probabilities sum to 1, i.e.,

$$\sum_{a \in A} f(a) = 1.$$

- A multiteam is a pair (X, m), where X is a set of assignments and $m : X \to \mathbb{N}^{>0}$ is a multiplicity function (a database with duplicates).
- A probabilistic team is a pair (X, p), where X is a set of assignments and $p : X \to \mathbb{Q}_{[0,1]}$ is a distribution (distribution of data).

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Probabilistic teams

- Modelling of data that is inherently a probability distribution.
- Abstraction of data with duplicates.
- There is close connection between multiteams and probabilistic teams.

We introduce a logic that describe properties of probabilistic teams.

We consider the expansion of first-order logic with the marginal identity atoms

$$(x_1, \ldots, x_n) \approx (y_1, \ldots, y_n)$$

and with the probabilistic conditional independence atoms

$$\overline{y} \perp\!\!\!\perp_{\overline{x}} \overline{z}.$$

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Probabilistic atoms

We define that

$$\mathfrak{A} \models_{\mathbb{X}} \vec{x} \approx \vec{y} \text{ iff the distribution of values for } \vec{x} \text{ and } \vec{y} \text{ in } \mathbb{X} \text{ coincide.}$$

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We define that

$$\mathfrak{A} \models_{\mathbb{X}} \overline{y} \perp\!\!\!\perp_{\overline{x}} \overline{z} \text{ iff for every fixed value for } \vec{x},$$

the value distribution of $\vec{y}$ remains unchanged if any value for $\vec{z}$ is given.

Probabilistic atoms

Let $\mathbb{X} = (X, p)$ be a probablistic team and $\vec{x}, \vec{a}$ be tuples of variables and values of length k. We define

$$|\mathbb{X}|_{\vec{x}=\vec{a}} := \sum_{\substack{s \in X \\ s(\vec{x})=\vec{a}}} p(s).$$

We define that

$$\mathfrak{A} \models_{\mathbb{X}} \vec{x} \approx \vec{y} \text{ iff } \; |\mathbb{X}|_{\vec{x}=\vec{a}} = |\mathbb{X}|_{\vec{y}=\vec{a}}, \text{ for each } \vec{a} \in A^k.$$

We define that

$$\mathfrak{A} \models_{\mathbb{X}} \overline{y} \perp\!\!\!\perp_{\overline{x}} \overline{z} \text{ iff for all assignments } s \text{ for } \vec{x}, \vec{y}, \vec{z}$$

$$|\mathbb{X}|_{\vec{x}\vec{y}=s(\vec{x}\vec{y})} \times |\mathbb{X}|_{\vec{x}\vec{z}=s(\vec{x}\vec{z})} = |\mathbb{X}|_{\vec{x}\vec{y}\vec{z}=s(\vec{x}\vec{y}\vec{z})} \times |\mathbb{X}|_{\vec{x}=s(\vec{x})}.$$

Semantics of complex formulae

Definition

Let $\mathfrak{A}$ be a structure over a finite domain A, and $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$ a probabilistic team of $\mathfrak{A}$. The satisfaction relation $\models_{\mathbb{X}}$ for first-order logic is defined as follows:

$$\mathfrak{A} \models_{\mathbb{X}} x = y \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(x) = s(y)$$

$$\mathfrak{A} \models_{\mathbb{X}} x \neq y \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(x) \neq s(y)$$

$$\mathfrak{A} \models_{\mathbb{X}} R(\overline{x}) \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(\overline{x}) \in R^{\mathfrak{A}}$$

$$\mathfrak{A} \models_{\mathbb{X}} \neg R(\overline{x}) \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(\overline{x}) \notin R^{\mathfrak{A}}$$

$$\mathfrak{A} \models_{\mathbb{X}} (\psi \wedge \theta) \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{X}} \theta$$

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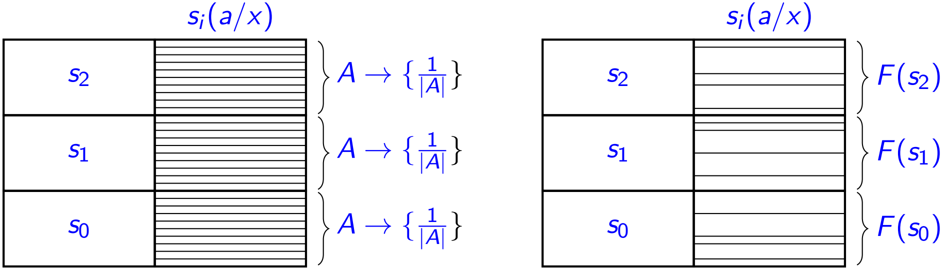
$$\mathfrak{A} \models_{\mathbb{X}} (\psi \vee \theta) \Leftrightarrow \mathfrak{A} \models_{\mathbb{Y}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{Z}} \theta \text{ for some } \mathbb{Y}, \mathbb{Z} \; s.t. \; \mathbb{Y} \sqcup \mathbb{Z} = \mathbb{X}$$

$$\mathfrak{A} \models_{\mathbb{X}} \forall x \psi \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}[A/x]} \psi$$

$$\mathfrak{A} \models_{\mathbb{X}} \exists x \psi \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}[F/x]} \psi \text{ holds for some } F\colon X \to p_A.$$

Above p_A denote the set those distributions that have domain A.

Intuition of the quantifiers



- Universal quantification (i.e., the set $\mathbb{X}[A/x]$) is depicted on left.
- Existential quantification (i.e., the set $\mathbb{X}[F/x]$) is depicted on right.
- Height of a box corresponds to the probability of an assignment.

Intuition behind the disjunction

Question: How do we split distributions?
Answer: We rescale.

Let $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$ and $\mathbb{Y}\colon Y \to \mathbb{Q}_{[0,1]}$ be probabilistic teams and $k \in \mathbb{Q}_{[0,1]}$ be a rational number.

We denote by $\mathbb{X} \sqcup_k \mathbb{Y}$ the k-scaled union of $\mathbb{X}$ and $\mathbb{Y}$, that is, the probabilistic team $\mathbb{X} \sqcup_k \mathbb{Y}\colon X \cup Y \to \mathbb{Q}_{[0,1]}$ defined s.t. for each $s \in X \cup Y$,

$$(\mathbb{X} \sqcup_k \mathbb{Y})(s) := \begin{cases} k \cdot \mathbb{X}(s) + (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in X \text{ and } s \in Y, \\ k \cdot \mathbb{X}(s) & \text{if } s \in X \text{ and } s \notin Y, \\ (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in Y \text{ and } s \notin X. \end{cases}$$

We then write that $Z = \mathbb{X} \sqcup \mathbb{Y}$ if $Z = \mathbb{X} \sqcup_k \mathbb{Y}$, for some k.

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Example

Consider a database table that lists results of experiments as a multiteam or as the related probabilistic team using the counting measure.

- Records: Outcomes of measurements obtained simultaneously in two locations.
- Attributes: Test1 and Test2 ranging over types of measurements, and Outcome1 and Outcome2 ranging over outcomes of the measurements.

The probabilistic independence atom $\mathrm{Test1} \perp\!\!\!\perp \mathrm{Test2}$ expresses that the types of measurements are independently picked in the two locations.

The marginal identity atom $(\mathrm{Test1}, \mathrm{Outcome1}) \approx (\mathrm{Test2}, \mathrm{Outcome2})$ expresses that the distributions of tests and results are the same in both test sites.

The formula $\mathrm{Test1} = \mathrm{Test2} \vee (\mathrm{Test1} \neq \mathrm{Test2} \wedge \mathrm{Outcome1} \perp\!\!\!\perp \mathrm{Outcome2})$ expresses that there is no correlation between outcomes of different measurements in the two test sites.

More examples

- The formula $\forall \vec{y}\, \vec{x} \approx \vec{y}$ states that the probabilities for $\overline{x}$ are uniformly distributed over all value sequences of length $|\overline{x}|$.
- The probability of $P(x)$ is at least twice the probability of $Q(x)$.
- Can we characterise the expressive power of $\mathrm{FO}(\approx, \perp\!\!\!\perp)$ in the probabilistic setting?

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- Can we characterise the expressive power of $\mathrm{FO}(\approx, \perp\!\!\!\perp)$ in the probabilistic setting?

Benchmark logic

- In team semantics context fragments of second-order logic are captured.
- $\mathrm{FO}(\perp)$ (team semantics) is as expressive as existential second-order logic.
- We define a two-sorted variant of ESO in which we allow the quantification of rational distributions.
- This logic characterises the expressive power of $\mathrm{FO}(\approx, \perp\!\!\!\perp)$.

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Probabilistic structures

Definition

Let τ and σ be a relational and a functional vocabulary. A probabilistic $\tau \cup \sigma$-structure is a tuple

$$\mathfrak{A} = (A, \mathbb{Q}_{[0,1]}, (R_i^{\mathfrak{A}})_{R_i \in \tau}, (f_i^{\mathfrak{A}})_{f_i \in \sigma}),$$

where

- A (i.e. the domain of $\mathfrak{A}$) is a finite nonempty set,
- $\mathbb{Q}_{[0,1]}$ is the set of rational numbers in the closed interval $[0, 1]$,
- each $R_i^{\mathfrak{A}}$ is a relation on A (i.e., a subset of $A^{\mathrm{ar}(R_i)}$),
- each $f_i^{\mathfrak{A}}$ is a probability distribution from $A^{\mathrm{ar}(f_i)}$ to $\mathbb{Q}_{[0,1]}$ (i.e., a function such that $\sum_{\vec{a} \in A^{\mathrm{ar}(f_i)}} f_i(\vec{a}) = 1$).

Second-order logic for probabilistic structures

- As first-order terms we have first-order variables.
- The set of numerical σ-terms i is defined via the grammar

$$i ::= f(\vec{x}) \mid i \times i \mid \mathrm{SUM}_{\vec{x}}\, i(\vec{x}, \vec{y}),$$

  where $\vec{x}, \vec{y}$ are tuples of first-order variables, $f \in \sigma$ and σ is a set of functions.
- The value of a numerical term i in a structure $\mathfrak{A}$ under an assignment s is denoted by $[i]_s^{\mathfrak{A}}$ and defined as follows:

$$[f(\overline{x})]_s^{\mathfrak{A}} := f^{\mathfrak{A}}(s(\overline{x})), \qquad [i \times j]_s^{\mathfrak{A}} := [i]_s^{\mathfrak{A}} \cdot [j]_s^{\mathfrak{A}},$$

$$[\mathrm{SUM}_{\vec{x}}\, i(\vec{x}, \vec{y})]_s^{\mathfrak{A}} := \sum_{\vec{a} \in A^{|\vec{x}|}} [i(\vec{a}, \vec{y})]_s^{\mathfrak{A}},$$

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Second-order logic for probabilistic structures

Definition

The formulae of ESOf$_{\mathbb{Q}}$ is defined via the following grammar:

$$\phi ::= x = y \mid x \neq y \mid i = j \mid i \neq j \mid R(\vec{x}) \mid \neg R(\vec{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi \mid \exists f \phi,$$

where i is a numerical term, R is a relation symbol, f is a function variable, $\vec{x}$ is a tuple of first-order variables.

Semantics of ESOf$_{\mathbb{Q}}$ is defined via probabilistic structures and assignments analogous to FO. In addition to the clauses of first-order logic, we have:

$$\mathfrak{A} \models_s i = j \Leftrightarrow [i]^{\mathfrak{A}}_s = [j]^{\mathfrak{A}}_s, \qquad \mathfrak{A} \models_s i \neq j \Leftrightarrow [i]^{\mathfrak{A}}_s \neq [j]^{\mathfrak{A}}_s,$$

$\mathfrak{A} \models_s \exists f \phi \Leftrightarrow \mathfrak{A}[h/f] \models_s \phi$ for some probability distribution $h \colon A^{\mathrm{ar}(f)} \to \mathbb{Q}_{[0,1]}$,

where $\mathfrak{A}[h/f]$ denotes the expansion of $\mathfrak{A}$ that interprets f to h.

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Translating from $\mathrm{FO}(\perp\!\!\!\perp, \approx)$ to $\mathrm{ESOf}_{\mathbb{Q}}$

For a probabilistic team $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$, we let $f_{\mathbb{X}} : A^n \to \mathbb{Q}_{[0,1]}$ be the probability distribution such that $f_{\mathbb{X}}(s(\overline{x})) = \mathbb{X}(s)$ for all $s \in X$.

Theorem

For every $\phi(\overline{x}) \in \mathrm{FO}(\perp\!\!\!\perp, \approx)$ there is a formula $\phi^*(f) \in \mathrm{ESOf}_{\mathbb{Q}}$ with one free function variable f s.t. for all structures $\mathfrak{A}$ and nonempty probabilistic teams $\mathbb{X}$

$$\mathfrak{A} \models_{\mathbb{X}} \phi(\overline{x}) \iff (\mathfrak{A}, f_{\mathbb{X}}) \models \phi^*(f)$$

and vice versa.

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