Descriptive complexity of real computation and probabilistic independence logic

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Logics of dependence and independence

Probabilistic independence logic is the extension of first-order logic with conditional independence

Defined as other modern logics for dependence and independence:

Base logic	New atoms
First-order	Dependence
Modal	Independence
Propositional	Inclusion

Historical predecessors: First-order logic + richer quantification of variables

- Partially ordered quantifiers [Henkin, 1961]
- Independence-friendly logic [Hintikka and Sandu, 1989]

Logics of dependence and independence

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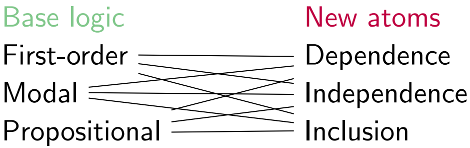

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Team semantics

Compositional semantics for complex dependence statements by team semantics [Hodges, 1997]

Team = set of objects (assignments, possible worlds, Boolean assignments)

`Employee`	`Department`	`Salary`
Alice	Math	50k
Bob	CS	40k
Carol	Physics	60k
David	Math	80k

New atoms = basic dependence statements about teams
(e.g, `Employee` determines `Salary`)

$\{\forall, \exists, \Box, \Diamond, \wedge, \vee\}$ for complex dependence statements

Reasoning about dependencies

Dependence and independence pivotal notions in many areas (databases, social choice, quantum foundations, ...)

Team logics can be used to express and formally prove results in these fields

- Arrow's theorem [Pacuit and Yang, 2016]
- Bell's theorem [Hyttinen et al., 2015]
- Implication problems for data dependencies [Hannula and Kontinen, 2016]

No "general" proof system: validity problem usually non-arithmetical.

Qualitative vs. quantitative dependence

Team logics can reason only about qualitative (relational) dependencies.

What about quantitative (probabilistic) dependencies?

Qualitative:

Functional dependency $X \to Y$

Multivalued dependency $X \twoheadrightarrow Y$

Inclusion dependency $X \subseteq Y$

Quantitative:

Marginal independence $X \perp\!\!\!\perp Y$

Conditional independence $X \perp\!\!\!\perp Y \mid Z$

Identical distribution of X and Y

Probabilistic team semantics

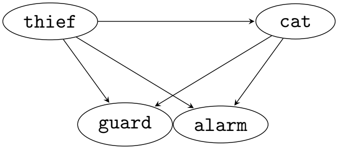
Basic concepts:

- Probabilistic team = probability distribution on a finite team [Durand et al., 2018]
- Quantitative atoms (e.g., conditional independence, identical distribution)
- $\{\forall, \exists, \wedge, \vee\}$ for complex probability statements

Probabilistic independence logic = first-order logic + conditional independence

Cf. recent probabilistic and quantitative approaches to separation logic
[Barthe et al., 2020, Batz et al., 2019]

Example



guard

thief,cat	T	F
TT	0.8	0.2
TF	0.7	0.3
FT	0	1
FF	0	1

thief

T	F
0.1	0.9

cat

thief	T	F
T	0.1	0.9
F	0.6	0.4

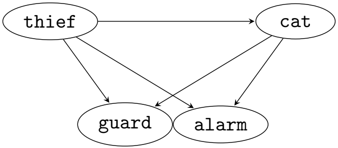
alarm

thief,cat	T	F
TT	0.9	0.1
TF	0.8	0.2
FT	0.1	0.9
FF	0	1

From the Bayesian network above we obtain that the joint probability distribution for t, c, g, a can be factorized as

$$P(t, c, g, a) = P(t) \cdot P(c \mid t) \cdot P(g \mid t, c) \cdot P(a \mid t, c)$$

Example



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thief	T	F
T	0.1	0.9
F	0.6	0.4

alarm

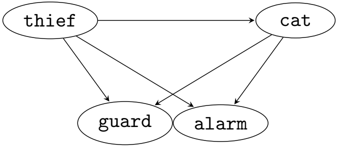
thief,cat	T	F
TT	0.9	0.1
TF	0.8	0.2
FT	0.1	0.9
FF	0	1

If additionally we have

$$\phi := t = F \to g = F$$

(i.e., guard never raises alert in absence of thief), the two bottom rows of the conditional probability table for guard become superfluous.

Example



guard

thief,cat	T	F
TT	0.8	0.2
TF	0.7	0.3
FT	0	1
FF	0	1

thief

T	F
0.1	0.9

cat

thief	T	F
T	0.1	0.9
F	0.6	0.4

alarm

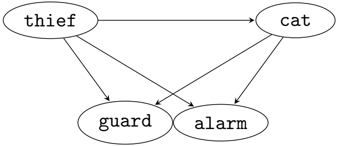
thief,cat	T	F
TT	0.8	0.2
TF	0.7	0.3
FT	0	1
FF	0	1

Given

$$\phi := tca \approx tcg$$

(i.e., conditioned on `thief` and `cat`, `alarm` and `guard` are identically distributed), then the conditional probability tables for `alarm` and `guard` are identical and one of them can be removed.

Example



guard

thief,cat	T	F
TT	0.45	0.55
TF	0.4	0.6
FT	0.05	0.95
FF	0	1

thief

T	F
0.1	0.9

cat

thief	T	F
T	0.1	0.9
F	0.6	0.4

alarm

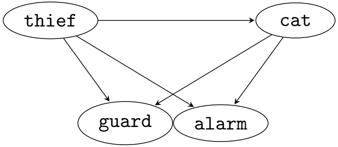
thief,cat	T	F
TT	0.9	0.1
TF	0.8	0.2
FT	0.1	0.9
FF	0	1

Given

$$\phi := \exists x(tcg \approx tcx \wedge tcga \perp\!\!\!\perp y \wedge x = T \leftrightarrow ay = TT)$$

(i.e., guard is of a factor $P(y = T)$ less sensitive to raise alert than alarm for any given thief and cat), it suffices to store the conditional probability table for alarm and the probability $P(y = T)$.

Example



$P(Y = T) = 0.5$

alarm

thief,cat	T	F
TT	0.9	0.1
TF	0.8	0.2
FT	0.1	0.9
FF	0	1

thief

T	F
0.1	0.9

cat

thief	T	F
T	0.1	0.9
F	0.6	0.4

Given

$$\phi := \exists x(tcg \approx tcx \wedge tcga \perp\!\!\!\perp y \wedge x = T \leftrightarrow ay = TT)$$

(i.e., `guard` is of a factor $P(y = T)$ less sensitive to raise alert than `alarm` for any given `thief` and `cat`), it suffices to store the conditional probability table for `alarm` and the probability $P(y = T)$.

Reasoning about probabilistic dependencies?

The implication problem for conditional independence $X \perp\!\!\!\perp Y \mid Z$

Input : A finite set $\Sigma \cup \{\sigma\}$ of CI statements
Output: Yes iff every finite probability distribution satisfying Σ satisfies also σ.

Theorem

The implication problem for conditional independence is:

(1) *in* Π^0_1 *[Khamis et al., 2020]*

(2) *in* EXPSPACE, *if restricted to binary domains [Hannula et al., 2019]*

- Decidability open
- Reduces to validity of probabilistic independence logic extended with classical negation; this problem is Π^0_1-complete

Probabilistic independence logic $\mathrm{FO}(\perp\!\!\!\perp_c)$

Syntax: FO (negation normal form) $+$ $\vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$ (only positively)

Semantics: Defined in terms of a finite structure $\mathfrak{A}$ and a probabilistic team $\mathbb{X}$

(1) Team $=$ a set of variable assignments with a shared domain

(2) Probabilistic team $=$ a pair $\mathbb{X} = (X, p)$, where X is a finite team and $p : X \to [0, 1]$ a probability distribution

Semantics of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$: probabilistic independence atoms

Let $\mathbb{X} = (X, p)$ be a probabilistic team and $\vec{x}, \vec{a}$ be tuples of variables and values.

$$|\mathbb{X}|_{\vec{x}=\vec{a}} := \sum_{\substack{s \in X \\ s(\vec{x})=\vec{a}}} p(s)$$

The semantics of probabilistic conditional independence atoms $\vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$:

$\mathfrak{A} \models_{\mathbb{X}} \vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$ iff, for all assignments s for $\vec{x}, \vec{y}, \vec{z}$

$$|\mathbb{X}|_{\vec{x}\vec{y}=s(\vec{x}\vec{y})} \cdot |\mathbb{X}|_{\vec{x}\vec{z}=s(\vec{x}\vec{z})} = |\mathbb{X}|_{\vec{x}\vec{y}\vec{z}=s(\vec{x}\vec{y}\vec{z})} \cdot |\mathbb{X}|_{\vec{x}=s(\vec{x})}.$$

Semantics of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$: the first-order part I

Definition ([Durand et al., 2018])

Let $\mathfrak{A}$ be a finite structure and $\mathbb{X} = (X, p)$ a probabilistic team.

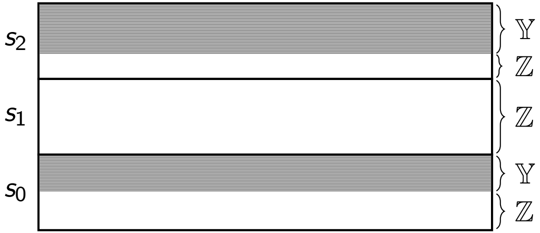
$$\mathfrak{A} \models_{\mathbb{X}} \ell \quad \Leftrightarrow \quad \mathfrak{A} \models_s \ell \text{ for all } s \in X \text{ such that } p(s) > 0$$
$$\text{(when } \ell \text{ is a first-order literal)}$$
$$\mathfrak{A} \models_{\mathbb{X}} (\psi \wedge \theta) \quad \Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{X}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{X}} \theta$$

Semantics of $\mathrm{FO}(\perp\!\!\!\perp_c)$: the first-order part II

Disjunction via convex combinations:

$$\mathfrak{A} \models_{\mathbb{X}} (\psi \vee \theta) \quad \Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{Y}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{Z}} \theta,$$

where $\mathbb{X} = \alpha \cdot \mathbb{Y} + (1-\alpha) \cdot \mathbb{Z}$, for some $\alpha \in [0,1]$.



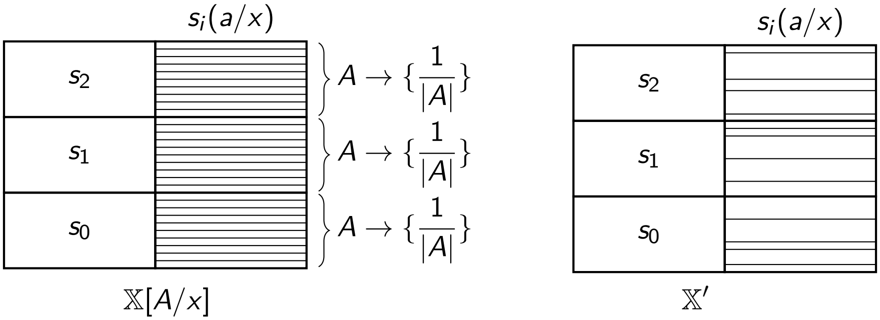
NB. The empty set is considered as a probabilistic team.

Semantics of $\mathrm{FO}(\perp\!\!\!\perp_c)$: the first-order part III

Quantification introduces a new column:

$$\mathfrak{A} \models_{\mathbb{X}} \forall x \psi \quad \Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{X}[A/x]} \psi$$

$$\mathfrak{A} \models_{\mathbb{X}} \exists x \psi \quad \Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{X}'} \psi, \text{ for some } \mathbb{X}' \text{ such that } \mathbb{X}' \upharpoonright \mathrm{Dom}(\mathbb{X}) = \mathbb{X}$$



Descriptive complexity

This paper: Determine the descriptive complexity of probabilistic independence logic

- Offers a machine independent description of complexity classes:
  - Time/Space used by a machine to decide a problem
    $\Rightarrow$ richness of the logical language needed to describe the problem.
- Complexity classes can/could be then separated by separating logics.
- Many characterisations are known:
  - Fagin's Theorem 1973: Existential second-order logic characterises NP.
  - Immerman & Vardi 1980s: Least fixed point logic LFP characterises P on ordered structures.

Descriptive complexity

This paper: Determine the descriptive complexity of probabilistic independence logic

Descriptive complexity in team logics:

1. Independence logic $\mathrm{FO}(\perp_{\mathrm{c}})$ equi-expressive to ESO $\implies$ captures NP.
2. Inclusion logic $\mathrm{FO}(\subseteq)$ equi-expressive to positive greatest fixed point-logic $\implies$ captures P on ordered structures [Galliani and Hella, 2013].

How to approach complexity in probabilistic team logics?

BSS model of computation

We consider **Blum-Shub-Smale machines** [Blum et al., 1989]

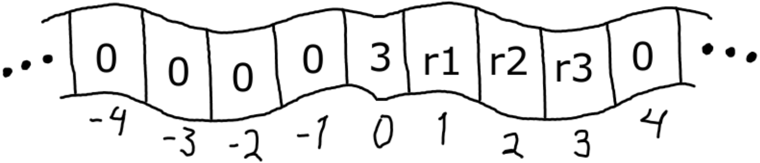
Input: finite string of reals, placed on bi-infinite tape $(\ldots, x_{-1}, x_0, x_1, \ldots)$
Output: 0 or 1 (decision problems)

A program is a finite list of instructions:

- Arithmetic instructions $x_i \leftarrow (x_j + x_k)$, $x_i \leftarrow (x_j - x_k)$, $x_i \leftarrow (x_j \times x_k)$, $x_i \leftarrow c$.
- Shift left or right.
- Branch on inequality, e.g., **if** $x_0 \leq 0$ **then** go to α; **else** go to β.

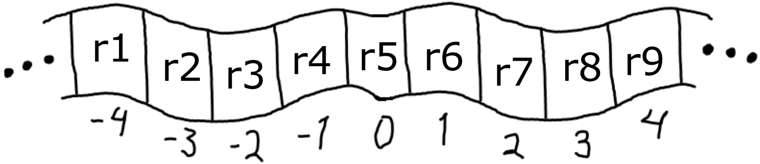
BSS instructions

Input: r1r2r3

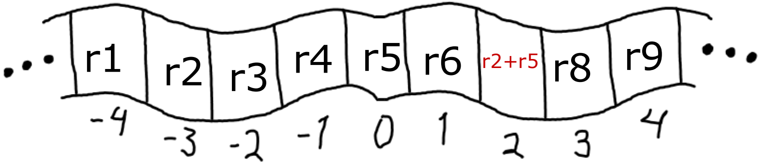


Output: r1

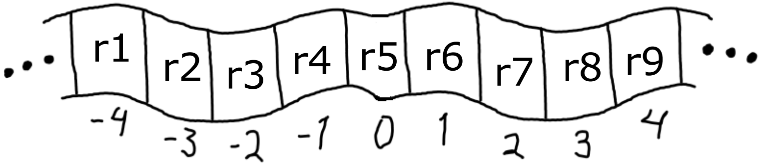
BSS instructions



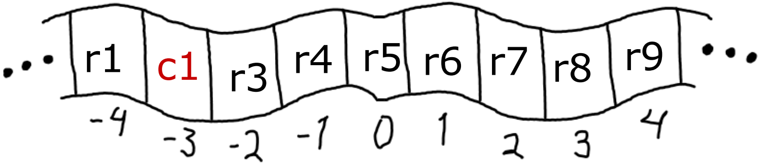
Addition: [2] := [-3]+[0]



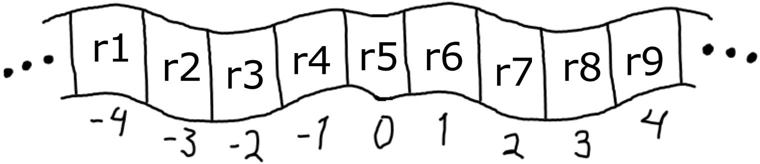
BSS instructions



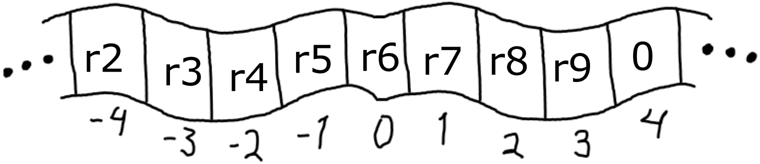
Assignment: [-3] := c1



BSS instructions



Shift left



Nondeterministic BSS

Nondeterminism is implemented by guessing a certificate:

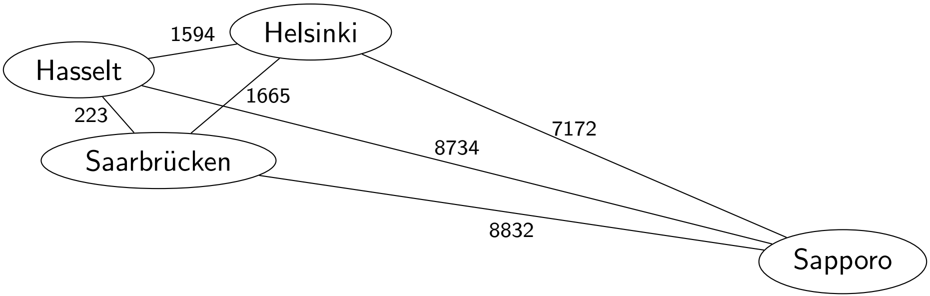
$\mathcal{L} \in \mathrm{NP}_{\mathbb{R}}$	there exists a BSS machine M s.t. $x \in \mathcal{L}$ iff $\exists y \in \mathbb{R}^*$ s.t. M accepts (x, y) in polynomial time in x

Example $\mathrm{NP}_{\mathbb{R}}$-complete problem: Is there a real root for a polynomial of degree 4?

BSS machines and logics on $\mathbb{R}$-structures

$\mathbb{R}$-structures [Grädel and Meer, 1995] consist of a finite structure $\mathfrak{A}$ together with an ordered field of reals and a finite set of weight functions from $\mathfrak{A}$ to $\mathbb{R}$.

(particular case of metafinite structures [Grädel and Gurevich, 1998])



BSS machines and logics on $\mathbb{R}$-structures cont.

Descriptive complexity w.r.t. $\mathbb{R}$-structures via BSS machines:

Theorem ([Grädel and Meer, 1995])

$\mathrm{ESO}_{\mathbb{R}}[+, \times, \leq, (r)_{r \in \mathbb{R}}] \equiv \mathrm{NP}_{\mathbb{R}}$

Two-sorted variant of ESO with

1. first-order logic on the finite structure $\mathfrak{A}$
2. existential quantification of functions from $\mathfrak{A}$ to reals
3. constants r for each real
4. complex numerical terms by $\{+, \times\}$
5. inequality $\leq$ between numerical terms

Too strong for $\mathrm{FO}(\perp\!\!\!\perp_c)$: 1) Lacks negation, 2) Quantification over $[0, 1]$

BSS machines and logics on $\mathbb{R}$-structures cont.

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- Arithmetic instructions $x_i \leftarrow (x_j + x_k)$, $x_i \leftarrow (x_j - x_k)$, $x_i \leftarrow (x_j \times x_k)$, $x_i \leftarrow c$
- Shift left or right
- Separate branch on inequality ($\epsilon^- < \epsilon^+$ are real numbers):

  if $x_0 \leq \epsilon^-$ **then** go to α;
  else if $x_0 \geq \epsilon^+$ **then** go to β;
  else reject.

Nondeterministic S-BSS

Nondeterminism here implemented by guessing a certificate from $[0,1]$:

$\mathcal{L} \in \text{S-NP}_{[0,1]}$ there exists an S-BSS machine M s.t. $x \in \mathcal{L}$ iff $\exists y \in [0,1]^*$ s.t. M accepts (x, y) in polynomial time in x

$\mathcal{L} \in \text{NP}_{\mathbb{R}}$ there exists a BSS machine M s.t. $x \in \mathcal{L}$ iff $\exists y \in \mathbb{R}^*$ s.t. M accepts (x, y) in polynomial time in x

Main result: $\mathrm{FO}(\perp\!\!\!\perp_c)$ and real computation

Descriptive complexity of $\mathrm{FO}(\perp\!\!\!\perp_c)$ in real computation:

Theorem

$\mathrm{FO}(\perp\!\!\!\perp_c) \equiv \mathrm{L}\text{-}\mathrm{ESO}_{[0,1]}[+, \times, \leq] \equiv \text{S-NP}^{0}_{[0,1]}$

- "Loose fragment": no negated atoms $\neg i \leq j$ between two numerical terms
- Existential second-order quantification over functions from $\mathrm{Dom}(\mathfrak{A})$ to $[0, 1]$
- Superscript 0: only machine constants 0 and 1 allowed

NB. The result holds for formulae of $\mathrm{FO}(\perp\!\!\!\perp_c)$

What is the relationship between $\text{S-NP}_{[0,1]}$ and $\mathrm{NP}_{\mathbb{R}}$?

Main result cont.: Separation of BSS and S-BSS computation

Theorem ([Blum et al., 1989])

Every language decidable by a (deterministic) BSS machine is a countable disjoint union of semi-algebraic sets.

Theorem

Every language decidable by

- *a deterministic S-BSS machine, or*
- *a time bounded $[0, 1]$-nondeterministic S-BSS machine*

is a countable disjoint union of closed sets in $\mathbb{R}^n$.

Main result cont.: Separation of BSS and S-BSS computation

Theorem

Every language decidable by

- *a deterministic S-BSS machine, or*
- *a time bounded* $[0,1]$*-nondeterministic S-BSS machine*

is a countable disjoint union of closed sets in $\mathbb{R}^n$.

Proof.

- The set of strings $s \in \mathbb{R}^n$ accepted by an S-BSS machine M in time (at most) t can be described by an $\mathrm{L\text{-}EFO}_{[0,1]}$ formula in $(\mathbb{R}, +, \times, \leq, 0, 1)$.
- Every n-ary relation defined by some $\mathrm{L\text{-}EFO}_{[0,1]}$ formula is closed in $\mathbb{R}^n$.

□

Main result cont.: Separation of BSS and S-BSS computation

Theorem

Every language decidable by

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is a countable disjoint union of closed sets in $\mathbb{R}^n$*.*

Theorem

$\text{S-NP}_{[0,1]} < \text{NP}_{\mathbb{R}}$

Main result: $\mathrm{FO}(\perp\!\!\!\perp_c)$ and real computation cont.

This separation holds also wrt. machines with constants $0, 1$

Descriptive complexity of $\mathrm{FO}(\perp\!\!\!\perp_c)$ thus strictly below $\mathsf{NP}^0_{\mathbb{R}}$:

Corollary

$\mathrm{FO}(\perp\!\!\!\perp_c) \equiv \text{S-}\mathsf{NP}^0_{[0,1]} < \mathsf{NP}^0_{\mathbb{R}}$

Scope of corollary: formulae of $\mathrm{FO}(\perp\!\!\!\perp_c)$

What about sentences of $\mathrm{FO}(\perp\!\!\!\perp_c)$?

Existential theory of the reals

- The existential theory of the reals consists of all true sentences of the form

$$\exists x_1, \ldots \exists x_n \psi(x_1, \ldots x_n)$$

  where ψ is a quantifier-free formula of the real arithmetic
- Gives rise to the Boolean complexity class $\exists\mathbb{R}$:
  the closure of the existential theory of the reals under polynomial-time reductions
- NP $\leq \exists\mathbb{R} \leq$ PSPACE
- Many natural geometric and algebraic problems are complete for $\exists\mathbb{R}$, such as the art gallery problem or recognition of unit distance graphs

Existential theory of the reals and BSS machines

Theorem ([Bürgisser and Cucker, 2006, Grädel and Meer, 1995, Schaefer and Stefankovic, 2017])

$\exists\mathbb{R} \equiv \mathrm{BP}(\mathsf{NP}^0_{\mathbb{R}}) \equiv \mathrm{ESO}_{\mathbb{R}}[+, \times, \leq]$

$\mathsf{NP}_{\mathbb{R}}$ restricted to Boolean inputs and with machine constants $0, 1$

Too strong for sentences of $\mathrm{FO}(\perp\!\!\!\perp_c)$?

Main result 2 – $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ and Boolean computation

Define $\exists[0,1]^{\leq}$ to be the fragment of $\exists\mathbb{R}$ obtained by closing the true sentences of the existential theory of the reals of the form

$$\exists x_1 \ldots \exists x_n \big(\bigwedge_{1 \leq i \leq n} 0 \leq x_i \wedge x_i \leq 1 \wedge \psi \big),$$

where ψ does not contain $\neg$ nor $<$, by polynomial-time reductions.

(Cf. $\mathrm{L\text{-}ESO}_{[0,1]}[+, \times \leq]$ vs. $\mathrm{ESO}_{\mathbb{R}}[+, \times \leq]$)

Theorem

Over finite structures, $\mathrm{FO}(\perp\!\!\!\perp) \equiv \exists[0,1]^{\leq}$.

Open question: Does $\exists[0,1]^{\leq}$ coincide with NP or $\exists\mathbb{R}$?

Main result 2 – $\mathrm{FO}(\perp\!\!\!\perp_c)$ and Boolean computation

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Theorem

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Conclusion

- Descriptive complexity of probabilistic independence logic $\mathrm{FO}(\perp\!\!\!\perp_{c})$
- We characterized $\mathrm{FO}(\perp\!\!\!\perp_{c})$
  - logically using a weakening of $\mathrm{ESO}_{\mathbb{R}}[+, \times, \leq]$
  - computationally using a novel S-BSS machine
- Over finite structures $\mathrm{FO}(\perp\!\!\!\perp)$ corresponds to a bounded fragment of the existential theory of the reals, $\exists[0,1]^{\leq}$
- S-BSS weaker than BSS: captures only unions of closed sets in $\mathbb{R}^n$
- Open questions:
  - Is $\exists[0,1]^{\leq}$ a distinct complexity class between NP and $\exists\mathbb{R}$?
  - Are there algebraic/geometric problems that are complete for $\exists[0,1]^{\leq}$?

Thanks!

(these slides are available at www.virtema.fi/slides/lics2020.pdf)

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