Complexity of Validity of Propositional Dependence Logics

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Motivation and history

Logical modelling of uncertainty, imperfect information and functional dependence in the framework of propositional (modal) logic.

The ideas are transfered from first-order dependence logic (and independence-friendly logic) to propositional (modal) logic.

Historical development:

- Branching quantifiers by Henkin 1959.
- Independence-friendly logic by Hintikka and Sandu 1989.
- Compositional semantics for independence-friendly logic by Hodges 1997. (Origin of team semantics.)
- IF modal logic by Tulenheimo 2003.
- Dependence logic by Väänänen 2007.
- Modal dependence logic by Väänänen 2008.

Syntax for propositional logics

Definition

Let Φ be a set of atomic propositions. The set of formulae for propositional logic $\mathcal{PL}(\Phi)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi),$$

where $p \in \Phi$.

The syntax for standard modal logic $\mathcal{ML}(\Phi)$ extends the syntax for $\mathcal{PL}(\Phi)$ by the grammar rules

$$\varphi ::= \Diamond\varphi \mid \Box\varphi.$$

Note that formulas are assumed to be in negation normal form: negations may occur only in front of atomic formulas.

Semantics for propositional logics

The semantics for $\mathcal{PL}(\Phi)$ and $\mathcal{ML}(\Phi)$ could be defined as usual, i.e., with assignments and pointed Kripke models, respectively.

In order to simplify the presentation, at this point, we consider propositional logic as a fragment of modal logic without modalities.

Definition

Let Φ be a set of atomic propositions. A Kripke model K over Φ is a tuple

$$K = (W, R, V),$$

where W is a nonempty set of *worlds*, $R \subseteq W \times W$ is a binary relation, and V is a *valuation* $V : \Phi \to \mathcal{P}(W)$.

We will give team semantics for $\mathcal{PL}(\Phi)$ and $\mathcal{ML}(\Phi)$.

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   (a) $K, w \models \varphi$: The actual world is w and φ is true in w.
   (b) $K, T \models \varphi$: The actual world is in T, but we do not know which one it is. The formula φ is true in the actual world.
   (c) $K, T \models \varphi$: We consider sets of points as primitive. The formula φ describes properties of collections of points.

Team semantics for modal logic

Definition

Kripke/Team semantics for $\mathcal{ML}$ is defined as follows. Remember that $K = (W, R, V)$ is a normal Kripke model and $T \subseteq W$.

$$
\begin{aligned}
K, w \models p \quad &\Leftrightarrow \quad w \in V(p).\\
K, w \models \neg p \quad &\Leftrightarrow \quad w \notin V(p).\\
K, w \models \varphi \wedge \psi \quad &\Leftrightarrow \quad K, w \models \varphi \text{ and } K, w \models \psi.\\
K, w \models \varphi \vee \psi \quad &\Leftrightarrow \quad K, w \models \varphi \text{ or } K, w \models \psi.\\
K, w \models \Box\varphi \quad &\Leftrightarrow \quad K, w' \models \varphi \text{ for every } w' \text{ s.t. } wRw'.\\
K, w \models \Diamond\varphi \quad &\Leftrightarrow \quad K, w' \models \varphi \text{ for some } w' \text{ s.t. } wRw'.
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K, T \models \neg p \quad &\Leftrightarrow \quad T \cap V(p) = \emptyset.\\
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K, T \models \Diamond\varphi &\quad\Leftrightarrow\quad K, T' \models \varphi \text{ for some } T' \text{ s.t.}\\
&\qquad\quad \forall w \in T\, \exists w' \in T' : wRw' \text{ and } \forall w' \in T'\, \exists w \in T : wRw'.
\end{aligned}
$$

Note that $K, \emptyset \models \varphi$ for every formula φ.

Team semantics vs. Kripke semantics

Theorem (Flatness property of ML)

Let K be a Kripke model, T a team of K and φ a $\mathcal{ML}$-formula. Then

$$K, T \models \varphi \quad \Leftrightarrow \quad K, w \models \varphi \text{ for all } w \in T,$$

in particular

$$K, \{w\} \models \varphi \quad \Leftrightarrow \quad K, w \models \varphi.$$

Modal dependence logic

Introduced by Väänänen 2008, the syntax modal dependence logic $\mathcal{MDL}$ extends the syntax of modal logic by the clause

$$\mathrm{dep}(p_1, \ldots, p_n, q),$$

where $p_1, \ldots, p_n, q$ are proposition symbols.

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Semantics for $\mathcal{MDL}$

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The semantics for $\mathcal{MDL}$ extends the sematics of $\mathcal{ML}$, defined with teams, by the following clause:

$$K, T \models \mathrm{dep}(p_1, \ldots, p_n, q)$$

if and only if $\forall w_1, w_2 \in T$:

$$\bigwedge_{i \leq n} \big(w_1 \in V(p_i) \Leftrightarrow w_2 \in V(p_i)\big) \Rightarrow \big(w_1 \in V(q) \Leftrightarrow w_2 \in V(q)\big).$$

Extended modal dependence logic $\mathcal{EMDL}$

$\mathcal{EMDL}(\Phi)$-formulas are defined by the following grammar:

$$\varphi \quad ::= \quad p \mid \neg p \mid \mathrm{dep}(\psi_1, \ldots, \psi_n, \theta) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \Phi$ and $\psi_1, \ldots, \psi_n, \theta \in \mathcal{ML}$.

The semantics of $\mathrm{dep}(\psi_1, \ldots, \psi_n, \theta)$ is given as for $\mathrm{dep}(p_1, \ldots, p_n, q)$.

With these more general dependence atoms we can express for example temporal dependencies.

Complexity results

	SAT	VAL	MC
$\mathcal{PL}$	NP [1]	coNP [1]	in P
$\mathcal{PD}$	NP [5]	??	NP [4]
$\mathcal{ML}$	PSPACE [2]	PSPACE [2]	in P
$\mathcal{MDL}$	NEXPTIME [3]	??	NP [4]
$\mathcal{EMDL}$	NEXPTIME [6]	??	NP [6]

[1] Cook 1971, Levin 1973, [2] Ladner 1977, [3] Sevenster 2009,
[4] Ebbing, Lohmann 2012, [5] Lohmann, Vollmer 2013,
[6] Ebbing, Hella, Meier, Müller, V., Vollmer 2013.

Team semantics for $\mathcal{PD}$ (and $\mathcal{PL}$)

Let Φ be a finite set of proposition symbols and let X be a set of assignments $s : \Phi \mapsto \{0, 1\}$. We call such an X a propositional team.

$$
\begin{aligned}
X \models p \quad &\Leftrightarrow \quad \forall s \in X : s(p) = 1.\\
X \models \neg p \quad &\Leftrightarrow \quad \forall s \in X : s(p) = 0.\\
X \models \varphi \wedge \psi \quad &\Leftrightarrow \quad X \models \varphi \text{ and } X \models \psi.\\
X \models \varphi \vee \psi \quad &\Leftrightarrow \quad Y \models \varphi \text{ and } Z \models \psi \text{ for some } Y \cup Z = X.\\
X \models \mathrm{dep}(p_1, \ldots, p_n, q) \quad &\Leftrightarrow \quad \forall s, t \in X : s(p_1) = t(p_1), \ldots, s(p_n) = t(p_n)\\
&\qquad \text{implies that } s(q) = t(q).
\end{aligned}
$$

Validity problem for $\mathcal{PD}$ is in NEXPTIME

For $\varphi \in \mathcal{PD}$, let $S(\varphi)$ denote the set of exactly all proposition symbols that occur in φ. Let $X_{S(\varphi)}$ denote the set of all assignments $s : S(\varphi) \mapsto \{0, 1\}$.

Proof.

- $\varphi \in \mathcal{PD}$ is valid iff $X_{S(\varphi)} \models \varphi$.
- The team $X_{S(\varphi)}$ can be clearly constructed from φ in exponential time.
- Checking whether $X_{S(\varphi)} \models \varphi$ can be done in NP in the combined size of $X_{S(\varphi)}$ and φ, and thus in NEXPTIME with respect to φ.

□

Validity problem for $\mathcal{PD}$ is NEXPTIME-hard

The proof uses a reduction from a NEXPTIME-complete variant of QBF called *Dependency quantified Boolean formulae* (DQBF) of Peterson, Reif, and Azhar 2001.

In the formulae of DQBF richer form of variable dependence can be expressed than in QBF. For example in the DQBF-formula

$$\forall\alpha_1\forall\alpha_2\exists\beta_1\exists\beta_2\psi, (\{\alpha_1\}, \{\alpha_2\})$$

the value for β_1 can depend only on the value of α_1, and the value for β_2 can depend only on the value of α_2.

Validity problem for $\mathcal{PD}$ is NEXPTIME-hard

The proof also uses the fact that $\varphi \in \mathcal{PD}$ is valid iff $X_{S(\varphi)} \models \varphi$.

Thus we get a prefix of universal quantification for free.

Disjunctions are used to simulate existential quantification and dependence atoms are used to uphold the wanted variable dependence.

Theorem

The validity problem for $\mathcal{PD}$ is NEXPTIME-complete.

Corollary

The validity problem for $\mathcal{MDL}$ and $\mathcal{EMDL}$ is NEXPTIME-hard.

The validity problem for $\mathcal{MDL}$ and $\mathcal{EMDL}$ is in $\mathrm{NEXPTIME}^{\mathrm{NP}}$

We have the following lemmas:

- Every $\varphi \in \mathcal{EMDL}$ is equivalent to some $\bigovee_{i \in I} \psi_i$, where each ψ_i is an exponential size $\mathcal{ML}$ formula and $\ovee$ intuitionistic disjunction.
- $\bigovee_{i \in I} \psi_i$ is valid iff ψ_i is valid for some $i \in I$.
- The decision problem whether a given $\mathcal{ML}$ formula is valid in small models is in $co\mathrm{NP}$.
- The ψ_is are such that ψ_i *is valid* iff ψ_i *is valid in small models*.

The validity problem for $\mathcal{MDL}$ and $\mathcal{EMDL}$ is in NEXPTIME$^{\mathrm{NP}}$

Proof.

An NEXPTIME$^{\mathrm{NP}}$ algorithm that checks whether $\varphi \in \mathcal{EMDL}$ is valid.

1. Guess nondeterministically an exponential size $\mathcal{ML}$ formula ψ.
2. Check whether ψ is among the ψ_is, $i \in I$. If not reject.
3. Use NP oracle to check whether ψ is valid in small models. Give the same output as the oracle.

□

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Thanks!