Polyteam Semantics

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Team Semantics: Motivation and History

Logical modelling of uncertainty, imperfect information, and different notions of dependence such as functional dependence and independence. Related to similar concepts in statistics, database theory etc.

Historical development:

- First-order logic and Skolem functions.
- Branching quantifiers by Henkin 1959.
- Independence-friendly logic by Hintikka and Sandu 1989.
- Compositional semantics for independence-friendly logic by Hodges 1997. (Origin of team semantics.)
- Dependence logic 2007 and modal dependence logic 2008 by Väänänen.
- Introduction of other dependency notions to team semantics such as inclusion, exclusion, and independence. Galliani, Grädel, Väänänen.
- Approximate dependence by Väänänen 2014 and multiteam semantics by Durand et al. 2016.

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First-Order Team Semantics (via database theoretic spectacles)

- A team is a set of assignments that have a common domain of variables.
- A team is a database table.
  - Variables correspond to attributes.
  - Assignments correspond to records.
- Dependency notions of database theory give rise to novel atomic formulae.
  - Functional dependence gives rise to dependence atoms $=(x_1, \ldots, x_n)$.
  - Inclusion dependence gives rise to inclusion atoms $\overline{x} \subseteq \overline{y}$.
  - Embedded multivalued dependency gives rise to independence atoms $\overline{y} \perp_{\overline{x}} \overline{z}$.

Dependence Logic

In FO, formulas are formed using connectives $\vee$, $\wedge$, $\neg$, and quantifiers $\exists$ and $\forall$.

Definition

Dependence logic $\mathrm{FO}(\mathrm{dep})$ extends the syntax of FO by dependence atoms

$$=\!(x_1, \dots, x_n)\,.$$

We consider also independence and inclusion atoms (and the corresponding logics) that replace dependence atoms respectively by

$$\overline{y} \perp_{\overline{x}} \overline{z} \text{ and } \overline{x} \subseteq \overline{y}.$$

Assignments and Teams

The semantics of dependence logic is defined using the notion of a team.

Teams:

Let A be a set and $V = \{x_1, \ldots, x_k, \ldots\}$ a set of variables. A *team* X with domain V is a set of assignments

$$s\colon V \to A.$$

A is called the co-domain of X (the universe of a model).

Interpretation of Dependence Atoms

Let $\mathfrak{A}$ be a structure and X a team with co-domain $\mathrm{Dom}(\mathfrak{A})$ and domain V s.t. $\{x_1, \ldots, x_n\} \subseteq V$.

$\mathfrak{A} \models_X =\!(x_1, \ldots, x_n)$, if and only if, for all $s, s' \in X$:

$$\bigwedge_{0<i<n} s(x_i) = s'(x_i) \implies s(x_n) = s'(x_n).$$

Interpreting Inclusion and Independence Atoms

Inclusion atoms:
$\mathfrak{A} \models_X \overline{x} \subseteq \overline{y}$, if and only if, for all $s \in X$ there exists $s' \in X$ s.t. $s(\overline{x}) = s'(\overline{y})$.

Independence atoms:
$\mathfrak{A} \models_X \overline{y} \bot_{\overline{x}} \overline{z}$, iff, for all $s, s' \in X$: if $s(\overline{x}) = s'(\overline{x})$ then there exists $s'' \in X$ such that

- $s''(\overline{x}) = s(\overline{x})$,
- $s''(\overline{y}) = s(\overline{y})$,
- $s''(\overline{z}) = s'(\overline{z})$.

Examples of teams

We may think of the variables x_i as attributes of a database such as $x_0 = \mathrm{SALARY}$ and $x_2 = \mathrm{ID\ NUMBER}$.

	x_0	.	.	.	x_n
s_0	$a_{0,m}$	.	.	.	$a_{n,m}$
.					
.					
.					
s_m	$a_{0,m}$	.	.	.	$a_{n,m}$

Then dependence atom $=\!(x_2, x_0)$ expresses the functional dependence

$$\mathrm{ID\ NUMBER} \rightarrow \mathrm{SALARY}.$$

Expressive Power

Dependence logic defines all downward closed ESO properties of teams.

Theorem (Kontinen, Väänänen 2009)

For every sentence $\psi \in \mathrm{ESO}[\tau \cup \{R\}]$*, in which* R *appears only negatively, there is* $\phi(y_1, \ldots, y_k) \in \mathrm{FO}(\mathrm{dep})[\tau]$ *s.t. for all* $\mathfrak{A}$ *and* $X \neq \emptyset$ *with domain* $\{y_1, \ldots, y_k\}$

$$\mathfrak{A} \models_X \phi \iff (\mathfrak{A}, R := X(\bar{y})) \models \psi.$$

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Theorem (Galliani 2012)

For every sentence $\psi \in \mathrm{ESO}[\tau \cup \{R\}]$ *there is* $\phi(y_1, \ldots, y_k) \in \mathrm{FO}(\perp)[\tau]$ *s.t. for all* $\mathfrak{A}$ *and* $X \neq \emptyset$ *with domain* $\{y_1, \ldots, y_k\}$*:*

$$\mathfrak{A} \models_X \phi \iff (\mathfrak{A}, R := X(\bar{y})) \models \psi.$$

Amstrong's Axioms for Functional Dependence

This inference system consists of only three rules which we depict below using the standard notation for functional dependencies, i.e., $X \to Y$ denotes that an attribute set X functionally determines another attribute set Y.

Definition (Armstrong 1974)

- Reflexivity: If $Y \subseteq X$, then $X \to Y$.
- Augmentation: if $X \to Y$, then $XZ \to YZ$
- Transitivity: if $X \to Y$ and $Y \to Z$, then $X \to Z$.

The same axiomatization works for dependence atoms $=(\overline{x}, y)$ when we add some rules that permutes and adds/removes duplicates to/from $\overline{x}$.

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Axioms for Pure (Marginal) Independence

For $\overline{x} \perp \overline{y}$, where $\overline{x}$ and $\overline{y}$ have no variables in common, a complete axiomatization is given by the following *Independence Axioms*:

1. Permutation and redundancy as before.
2. $\overline{x} \perp \emptyset$ (Empty Set Rule).
3. If $\overline{x} \perp \overline{y}$, then $\overline{y} \perp \overline{x}$ (Symmetry Rule).
4. If $\overline{x} \perp \overline{yz}$, then $\overline{x} \perp \overline{y}$ (Weakening Rule)
5. If $\overline{x} \perp \overline{y}$ and $\overline{xy} \perp \overline{z}$, then $\overline{x} \perp \overline{yz}$ (Exchange Rule).

This axiomatization due to Geiger, Paz, and Pearl (1991) for marginal independence $X \perp\!\!\!\perp Y$ between two sets of random variables.

From Teams to Polyteams

- Team semantics is a framework well suited to express different dependency notions, e.g., studied in database theory, when restricted to the unirelational case.
- However depedencies between different tables cannot be expressed in this framework. This is a real shortcoming.
- We next define a generalisation of team semantics in which we replace teams by tuples of teams.

Polyteams

For $i \in \mathbb{N}$, let $\mathrm{Var}(i)$ denote a distinct countable set of FO variable symbols.

Definition

Let $\mathfrak{A}$ be a τ-model and let $D_i \subseteq \mathrm{Var}(i)$ for all $i \in \mathbb{N}$. A tuple $\overline{X} = (X_i)_{i \in \mathbb{N}}$ is a *polyteam* of $\mathfrak{A}$ with domain $\overline{D} = (D_i)_{i \in \mathbb{N}}$, if X_i is a team with domain D_i and co-domain A for each $i \in \mathbb{N}$.

We identify $\overline{X}$ with $(X_1, \ldots, X_n)$ if X_i is empty for all i greater than n.
We write x^i, y^i, $\overline{x}^i$, etc., to denote variables from $\mathrm{Var}(i)$.

Polyatoms

Poly-Inclusion atoms:
$\mathfrak{A} \models_{\mathbb{X}} \overline{x}^i \subseteq \overline{y}^j$, iff, for all $s \in X_i$ there exists $s' \in X_j$ s.t. $s(\overline{x}^i) = s'(\overline{y}^j)$.

Poly-Dependence atoms:
Let $\overline{x}^i\overline{y}^i$ and $\overline{u}^j\overline{v}^j$ be sequences of variables s.t. $|\overline{x}^i| = |\overline{u}^j|$ and $|\overline{y}^i| = |\overline{u}^j|$.
Assume $i \neq j$.

$$\mathfrak{A} \models_{\mathbb{X}} =(\overline{x}^i, \overline{y}^i / \overline{u}^j, \overline{v}^j) \Leftrightarrow \forall s \in X_i \forall s' \in X_j : s(\overline{x}^i) = s'(\overline{u}^j) \text{ implies } s(\overline{y}^i) = s'(\overline{v}^j).$$

Note that the atom $=(\overline{x}, \overline{y} / \overline{x}, \overline{y})$ corresponds to the dependence atom $=(\overline{x}, \overline{y})$.
For empty tuples $\overline{x}^i$ and $\overline{u}^j$ the poly-dependence atom reduces to a "poly-constancy atom" $=(\overline{y}^i / \overline{v}^j)$.

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Assume $i \neq j$.

$$\mathfrak{A} \models_{\overline{X}} {=}(\overline{x}^i, \overline{y}^i / \overline{u}^j, \overline{v}^j) \Leftrightarrow \forall s \in X_i \forall s' \in X_j : s(\overline{x}^i) = s'(\overline{u}^j) \text{ implies } s(\overline{y}^i) = s'(\overline{v}^j).$$

Note that the atom ${=}(\overline{x}, \overline{y} / \overline{x}, \overline{y})$ corresponds to the dependence atom ${=}(\overline{x}, \overline{y})$.
For empty tuples $\overline{x}^i$ and $\overline{u}^j$ the poly-dependence atom reduces to a "poly-constancy atom" ${=}(\overline{y}^i / \overline{v}^j)$.

Axioms of Poly-Dependence

Definition (Axiomatization for poly-dependence atoms)

- Reflexivity: $=(\overline{x}^i, \mathrm{pr}_k(\overline{x}^i)/\overline{y}^j, \mathrm{pr}_k(\overline{y}^j))$, where $k = 1, \ldots, |\overline{x}^i|$ and pr_k takes the kth projection of a sequence.
- Augmentation: if $=(\overline{x}^i, \overline{y}^i/\overline{u}^j, \overline{v}^j)$, then $=(\overline{x}^i\overline{z}^i, \overline{y}^i\overline{z}^i/\overline{u}^j\overline{w}^j, \overline{v}^j\overline{w}^j)$
- Transitivity: if $=(\overline{x}^i, \overline{y}^i/\overline{u}^j, \overline{v}^j)$ and $=(\overline{y}^i, \overline{z}^i/\overline{v}^j, \overline{w}^j)$, then $=(\overline{x}^i, \overline{z}^i/\overline{u}^j, \overline{w}^j)$
- Union: if $=(\overline{x}^i, \overline{y}^i/\overline{u}^j, \overline{v}^j)$ and $=(\overline{x}^i, \overline{z}^i/\overline{u}^j, \overline{w}^j)$ then $=(\overline{x}^i, \overline{y}^i\overline{z}^i/\overline{u}^j, \overline{v}^j\overline{w}^j)$
- Symmetry: if $=(\overline{x}^i, \overline{y}^i/\overline{u}^j, \overline{v}^j)$, then $=(\overline{u}^j, \overline{v}^j/\overline{x}^i, \overline{y}^i)$
- Weak Transitivity: if $=(\overline{x}^i, \overline{y}^i\overline{z}^i\overline{z}^i/\overline{u}^j, \overline{v}^j\overline{v}^j\overline{w}^j)$, then $=(\overline{x}^i, \overline{y}^i/\overline{u}^j, \overline{w}^j)$

This proof system forms a complete characterization of logical implication for poly-dependence atoms.

Poly-Independence Atom

Independence atoms:
$\mathfrak{A} \models_X \overline{y} \perp_{\overline{x}} \overline{z}$, iff, for all $s, s' \in X$: if $s(\overline{x}) = s'(\overline{x})$ then there exists $s'' \in X$ such that

- $s''(\overline{x}) = s(\overline{x})$,
- $s''(\overline{y}) = s(\overline{y})$,
- $s''(\overline{z}) = s'(\overline{z})$.

Poly-Independence Atom

Poly-Independence atoms:
Let $\overline{x}^i$, $\overline{y}^i$, $\overline{a}^j$,$\overline{b}^j$, $\overline{u}^k$, $\overline{v}^k$, and $\overline{w}^k$ be tuples of variables such that $|\overline{x}^i| = |\overline{a}^j| = |\overline{u}^k|$, $|\overline{y}^i| = |\overline{v}^k|$, $|\overline{b}^j| = |\overline{w}^k|$.

$\mathfrak{A} \models_{\overline{X}} \overline{y}^i/\overline{v}^k \perp_{\overline{x}^i,\overline{a}^j/\overline{u}^k} \overline{b}^j/\overline{w}^k$, iff, for all $s \in X_i, s' \in X_j$: if $s(\overline{x}^i) = s'(\overline{a}^j)$ then there exists $s'' \in X_k$ such that

- $s''(\overline{u}^k) = s(\overline{x}^i)$,
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Poly-Independence atoms:

Let $\overline{x}^i$, $\overline{y}^i$, $\overline{a}^j$,$\overline{b}^j$, $\overline{u}^k$, $\overline{v}^k$, and $\overline{w}^k$ be tuples of variables such that $|\overline{x}^i| = |\overline{a}^j| = |\overline{u}^k|$, $|\overline{y}^i| = |\overline{v}^k|$, $|\overline{b}^j| = |\overline{w}^k|$.

$\mathfrak{A} \models_X \overline{y}^i/\overline{v}^k \perp_{\overline{x}^i,\overline{a}^j/\overline{u}^k} \overline{b}^j/\overline{w}^k$, iff, for all $s \in X_i, s' \in X_j$: if $s(\overline{x}^i) = s'(\overline{a}^j)$ then there exists $s'' \in X_k$ such that

- $s''(\overline{u}^k) = s(\overline{x}^i)$,
- $s''(\overline{v}^k) = s(\overline{y}^i)$,
- $s''(\overline{w}^k) = s'(\overline{b}^j)$.

The atom $\overline{y}/\overline{y} \perp_{\overline{x},\overline{x}/\overline{x}} \overline{z}/\overline{z}$ is the standard independence atom $\overline{y} \perp_{\overline{x}} \overline{z}$.

Desired Properties of Polyteam Semantics

- Let $\phi \in \mathrm{FO}$.
  For every team X it holds that $\mathfrak{A} \models_X \phi$ iff $\mathfrak{A} \models_s \phi$, for every $s \in X$.
- Let $\phi \in \mathrm{FO}$ whose variables are all of sort $i \in \mathbb{N}$.
  For every poly-team $\overline{X}$ it holds that $\mathfrak{A} \models_{\overline{X}} \phi$ iff $\mathfrak{A} \models_{X_i} \phi$.
- Let $\mathcal{L}$ be a team-based logic and $\phi \in \mathcal{L}$ whose variables are all of sort $i \in \mathbb{N}$.
  For every poly-team $\overline{X}$ it holds that $\mathfrak{A} \models_{\overline{X}} \phi$ iff $\mathfrak{A} \models_{X_i} \phi$.

Definition (Polyteam semantics for poly-first-order logic PFO)

Let $\mathfrak{A}$ be a τ-structure and $\overline{X}$ a polyteam of $\mathfrak{A}$. The satisfaction relation $\models_{\overline{X}}$ for first-order logic is defined as follows:

$$\begin{aligned}
\mathfrak{A} \models_{\overline{X}} x = y \quad &\Leftrightarrow \text{ if } x, y \in \mathrm{Var}(i) \text{ then } \forall s \in X_i : s(x) = s(y)\\
\mathfrak{A} \models_{\overline{X}} x \neq y \quad &\Leftrightarrow \text{ if } x, y \in \mathrm{Var}(i) \text{ then } \forall s \in X_i : s(x) \neq s(y)\\
\mathfrak{A} \models_{\overline{X}} R(\overline{x}) \quad &\Leftrightarrow \text{ if } \overline{x} \in \mathrm{Var}(i)^k \text{ then } \forall s \in X_i : s(\overline{x}) \in R^{\mathfrak{A}}\\
\mathfrak{A} \models_{\overline{X}} \neg R(\overline{x}) \quad &\Leftrightarrow \text{ if } \overline{x} \in \mathrm{Var}(i)^k \text{ then } \forall s \in X_i : s(\overline{x}) \notin R^{\mathfrak{A}}
\end{aligned}$$

Definition (Polyteam semantics for poly-first-order logic PFO)

Let $\mathfrak{A}$ be a τ-structure and $\overline{X}$ a polyteam of $\mathfrak{A}$. The satisfaction relation $\models_{\overline{X}}$ for first-order logic is defined as follows:

$$\mathfrak{A} \models_{\overline{X}} (\psi \wedge \theta) \Leftrightarrow \mathfrak{A} \models_{\overline{X}} \psi \text{ and } \mathfrak{A} \models_{\overline{X}} \theta$$

$$\mathfrak{A} \models_{\overline{X}} (\psi \vee \theta) \Leftrightarrow \mathfrak{A} \models_{\overline{Y}} \psi \text{ and } \mathfrak{A} \models_{\overline{Z}} \theta \text{ for some } \overline{Y}, \overline{Z} \subseteq \overline{X} \text{ s.t. } \overline{Y} \cup \overline{Z} = \overline{X}$$

$$\mathfrak{A} \models_{\overline{X}} (\psi \vee^j \theta) \Leftrightarrow \mathfrak{A} \models_{\overline{X}[Y_j/X_j]} \psi \text{ and } \mathfrak{A} \models_{\overline{X}[Z_j/X_j]} \theta$$

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$$\mathfrak{A} \models_{\overline{X}} \forall x\psi \quad \Leftrightarrow \mathfrak{A} \models_{\overline{X}[X_i[A/x]/X_i]} \psi, \text{ when } x \in \mathrm{Var}(i)$$

$$\mathfrak{A} \models_{\overline{X}} \exists x\psi \quad \Leftrightarrow \mathfrak{A} \models_{\overline{X}[X_i[F/x]/X_i]} \psi \text{ holds for some } F\colon X_i \to \mathcal{P}(A) \setminus \{\emptyset\}, \text{ when } x \in \mathrm{Var}(i).$$

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\mathfrak{A} \models_{\overline{X}} R(\overline{x}) &\Leftrightarrow \text{if } \overline{x} \in \mathrm{Var}(i)^k \text{ then } \forall s \in X_i : s(\overline{x}) \in R^{\mathfrak{A}} \\
\mathfrak{A} \models_{\overline{X}} \neg R(\overline{x}) &\Leftrightarrow \text{if } \overline{x} \in \mathrm{Var}(i)^k \text{ then } \forall s \in X_i : s(\overline{x}) \notin R^{\mathfrak{A}} \\
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\mathfrak{A} \models_{\overline{X}} \exists x \psi &\Leftrightarrow \mathfrak{A} \models_{\overline{X}[X_i[F/x]/X_i]} \psi \text{ holds for some } F : X_i \to \mathcal{P}(A) \setminus \{\emptyset\}, \\
&\qquad \text{when } x \in \mathrm{Var}(i).
\end{aligned}
$$

Relationship Between $\vee$ and $\vee^i$

For every formula of PFO there exists an equivalent formula of PFO that only uses disjunctions of type $\vee^i$.

Proof:
Guess the splits of $\vee$ by using quantifiers and then use $\vee_i$ for splitting each team one-by-one.

Polydependence Logics

- Add ordinary atoms (dependence, inclusion, etc.) to PFO.
- Add poly-atoms (poly-dependence, poly-inclusion, etc.) to PFO.
- The latter yields strictly more expressive logics.
- The poly-constancy atom $=(x^1/x^2)$ cannot be expressed in $\mathrm{PFO}(\mathrm{dep})$.

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Expressive Power of Poly-Dependence Logic

$\mathrm{FO}(\mathrm{dep})$ defines all downward closed ESO properties of teams.

Theorem (Kontinen, Väänänen 2009)

For every sentence $\psi \in \mathrm{ESO}[\tau \cup \{R\}]$, *in which* R *appears only negatively, there is* $\phi(y_1, \ldots, y_k) \in \mathrm{FO}(\mathrm{dep})[\tau]$ *s.t. for all* $\mathfrak{A}$ *and* $X \neq \emptyset$ *with domain* $\{y_1, \ldots, y_k\}$

$$\mathfrak{A} \models_X \phi \iff (\mathfrak{A}, R := X(\overline{y})) \models \psi.$$

$\mathrm{PFO}(\mathrm{dep})$ defines all conjunctions of downward closed ESO properties of teams.

Theorem

For every finite sequence ψ_i, $1 \leq i \leq n$, *of* $\mathrm{ESO}[\tau \cup \{R_i\}]$*-sentences, in which* R_i *appears only negatively, there is* $\phi(\overline{x}^1, \ldots, \overline{x}^n) \in \mathrm{PFO}(\mathrm{dep})[\tau]$ *s.t. for all* $\mathfrak{A}$ *and for all polyteams* $\overline{X} = (X_1, \ldots, X_n)$ *where* $\mathrm{Dom}(X_i) = \overline{x}^i$ *and* $X_i \neq \emptyset$

$$\mathfrak{A} \models_{\overline{X}} \phi(\overline{x}^1, \ldots, \overline{x}^n) \Leftrightarrow (\mathfrak{A}, R_1 := X_1(\overline{x}_1) \ldots, X_n(\overline{x}_n)) \models \psi_1(R_1) \wedge \ldots \wedge \psi_n(R_n).$$

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$\mathrm{PFO}(\mathrm{pdep})$ defines all downward closed ESO properties of polyteams.

Theorem

For every sentence $\psi \in \mathrm{ESO}[\tau \cup \{R_1, \ldots, R_n\}]$ in which R_i:s appear only negatively, there is $\phi(\overline{x}^1, \ldots, \overline{x}^n) \in \mathrm{PFO}(\mathrm{pdep})[\tau]$ s.t. for all $\mathfrak{A}$ and for all polyteams $\overline{X} = (X_1, \ldots, X_n)$ where $\mathrm{Dom}(X_i) = \overline{x}^i$ and $X_i \neq \emptyset$

$$\mathfrak{A} \models_{\overline{X}} \phi(\overline{x}^1, \ldots, \overline{x}^n) \Leftrightarrow (\mathfrak{A}, R_1 := X_1(\overline{x}_1) \ldots, X_n(\overline{x}_n)) \models \psi(R_1, \ldots, R_n).$$

Further Results and Details

- Results of the previous page also works for independence logics:
  - $\mathrm{PFO}(\mathrm{ind})$ defines all conjunctions of ESO properties of teams.
  - $\mathrm{PFO}(\mathrm{pind})$ defines all ESO properties of polyteams.
- For details see the preprint (Polyteam Semantics) in ArXiv.

THANKS!

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