Probabilistic Team Semantics

Jonni Virtema

Hasselt University, Belgium
jonni.virtema@gmail.com

Joint work with Arnaud Durand (Université Paris Diderot), Miika Hannula (University of Auckland), Juha Kontinen (University of Helsinki), and Arne Meier (Leibniz Universität Hannover)

May 17, 2018

Distributions

Consider:

- A collection of data from some repetitive science experiment.
- Data obtained from a poll.
- Any collection of data, that involves meaningful duplicates of data.

One natural way to represent the data is to use multisets (sets with duplicates).

Often the multiplicities themselves are not important; the distribution of data is:

- The locations of the electrons of an atom.
- Pre-election poll of party support.
- Distribution of a population with attributes like education, salary, and age.

Distributions

Consider:

- A collection of data from some repetitive science experiment.
- Data obtained from a poll.
- Any collection of data, that involves meaningful duplicates of data.

One natural way to represent the data is to use multisets (sets with duplicates).

Often the multiplicities themselves are not important; the distribution of data is:

- The locations of the electrons of an atom.
- Pre-election poll of party support.
- Distribution of a population with attributes like education, salary, and age.

Distributions

Definition

A distribution is a mapping $f : A \to \mathbb{Q}_{[0,1]}$ from a set A of values to the closed interval $[0, 1]$ of rational numbers such that the probabilities sum to 1, i.e.,

$$\sum_{a \in A} f(a) = 1.$$

- A team is a set of first-order assignments (a database without duplicates).
- A multiteam is a pair (X, m), where X is a team and $m : X \to \mathbb{N}^{>0}$ is a multiplicity function (a database with duplicates).
- A probabilistic team is a pair (X, p), where X is a team and $p : X \to \mathbb{Q}_{[0,1]}$ is a distribution (distribution of data).

Distributions

Definition

A distribution is a mapping $f : A \to \mathbb{Q}_{[0,1]}$ from a set A of values to the closed interval $[0, 1]$ of rational numbers such that the probabilities sum to 1, i.e.,

$$\sum_{a \in A} f(a) = 1.$$

- A team is a set of first-order assignments (a database without duplicates).
- A multiteam is a pair (X, m), where X is a team and $m : X \to \mathbb{N}^{>0}$ is a multiplicity function (a database with duplicates).
- A probabilistic team is a pair (X, p), where X is a team and $p : X \to \mathbb{Q}_{[0,1]}$ is a distribution (distribution of data).

Probabilistic teams

- Modelling of data that is inherently a probability distribution.
- Abstraction of data with duplicates.
- There is close connection between multiteams and probabilistic teams.

We introduce a logic that describe properties of probabilistic teams.

We consider the expansion of first-order logic with the marginal identity atoms

$$(x_1, \ldots, x_n) \approx (y_1, \ldots, y_n)$$

and with the probabilistic conditional independence atoms

$$\overline{y} \perp\!\!\!\perp_{\overline{x}} \overline{z}.$$

Probabilistic teams

- Modelling of data that is inherently a probability distribution.
- Abstraction of data with duplicates.
- There is close connection between multiteams and probabilistic teams.

We introduce a logic that describe properties of probabilistic teams.

We consider the expansion of first-order logic with the marginal identity atoms

$$(x_1, \ldots, x_n) \approx (y_1, \ldots, y_n)$$

and with the probabilistic conditional independence atoms

$$\overline{y} \perp\!\!\!\perp_{\overline{x}} \overline{z}.$$

Probabilistic atoms

The semantics are inherited from multiteam semantics.

Let $\mathbb{X} = (X, p)$ be a probablistic team and $\vec{x}, \vec{a}$ be tuples of variables and values of length k. We define

$$|\mathbb{X}|_{\vec{x}=\vec{a}} := \sum_{\substack{s \in X \\ s(\vec{x})=\vec{a}}} p(s).$$

Probabilistic atoms

Let $\mathbb{X} = (X, p)$ be a probablistic team and $\vec{x}, \vec{a}$ be tuples of variables and values of length k. We define

$$|\mathbb{X}|_{\vec{x}=\vec{a}} := \sum_{\substack{s \in X \\ s(\vec{x})=\vec{a}}} p(s).$$

We define that

$$\mathfrak{A} \models_{\mathbb{X}} \vec{x} \approx \vec{y} \text{ iff } |\mathbb{X}|_{\vec{x}=\vec{a}} = |\mathbb{X}|_{\vec{y}=\vec{a}}, \text{ for each } \vec{a} \in A^k,$$

Probabilistic atoms

Let $\mathbb{X} = (X, p)$ be a probablistic team and $\vec{x}, \vec{a}$ be tuples of variables and values of length k. We define

$$|\mathbb{X}|_{\vec{x}=\vec{a}} := \sum_{\substack{s \in X \\ s(\vec{x})=\vec{a}}} p(s).$$

We define that $\mathfrak{A} \models_{\mathbb{X}} \overline{y} \perp\!\!\!\perp_{\overline{x}} \overline{z}$ iff, for all assignments s for $\vec{x}, \vec{y}, \vec{z}$

$$|\mathbb{X}|_{\vec{x}\vec{y}=s(\vec{x}\vec{y})} \times |\mathbb{X}|_{\vec{x}\vec{z}=s(\vec{x}\vec{z})} = |\mathbb{X}|_{\vec{x}\vec{y}\vec{z}=s(\vec{x}\vec{y}\vec{z})} \times |\mathbb{X}|_{\vec{x}=s(\vec{x})}.$$

Semantics of complex formulae

Definition

Let $\mathfrak{A}$ be a structure over a finite domain A, and $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$ a probabilistic team of $\mathfrak{A}$. The satisfaction relation $\models_{\mathbb{X}}$ for first-order logic is defined as follows:

$$\mathfrak{A} \models_{\mathbb{X}} x = y \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(x) = s(y)$$
$$\mathfrak{A} \models_{\mathbb{X}} x \neq y \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(x) \neq s(y)$$
$$\mathfrak{A} \models_{\mathbb{X}} R(\overline{x}) \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(\overline{x}) \in R^{\mathfrak{A}}$$
$$\mathfrak{A} \models_{\mathbb{X}} \neg R(\overline{x}) \Leftrightarrow \text{for all } s \in X : \text{if } \mathbb{X}(s) > 0, \text{ then } s(\overline{x}) \notin R^{\mathfrak{A}}$$
$$\mathfrak{A} \models_{\mathbb{X}} (\psi \wedge \theta) \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{X}} \theta$$

Semantics of complex formulae

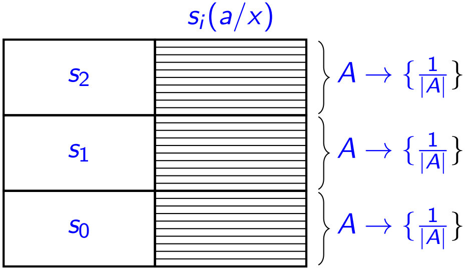
Definition

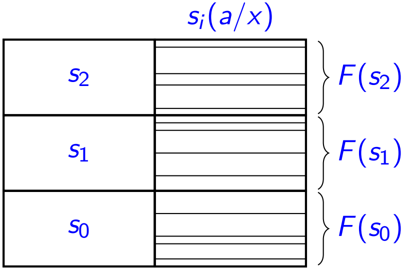
Let $\mathfrak{A}$ be a structure over a finite domain A, and $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$ a probabilistic team of $\mathfrak{A}$. The satisfaction relation $\models_{\mathbb{X}}$ for first-order logic is defined as follows:

$$\mathfrak{A} \models_{\mathbb{X}} (\psi \vee \theta) \Leftrightarrow \mathfrak{A} \models_{\mathbb{Y}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{Z}} \theta \text{ for some } \mathbb{Y}, \mathbb{Z} \ s.t. \ \mathbb{Y} \sqcup \mathbb{Z} = \mathbb{X}$$
$$\mathfrak{A} \models_{\mathbb{X}} \forall x \psi \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}[A/x]} \psi$$
$$\mathfrak{A} \models_{\mathbb{X}} \exists x \psi \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}[F/x]} \psi \text{ holds for some } F\colon X \to p_A.$$

Above p_A denote the set those distributions that have domain A.

Intuition of the quantifiers





- Universal quantification (i.e., the set $\mathbb{X}[A/x]$) is depicted on left.
- Existential quantification (i.e., the set $\mathbb{X}[F/x]$) is depicted on right.
- Height of a box corresponds to the probability of an assignment.

Intuition behind the disjunction

Question: How do we split distributions?
Answer: We rescale.

Let $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$ and $\mathbb{Y}\colon Y \to \mathbb{Q}_{[0,1]}$ be probabilistic teams and $k \in \mathbb{Q}_{[0,1]}$ be a rational number.

We denote by $\mathbb{X} \sqcup_k \mathbb{Y}$ the k-scaled union of $\mathbb{X}$ and $\mathbb{Y}$, that is, the probabilistic team $\mathbb{X} \sqcup_k \mathbb{Y}\colon X \cup Y \to \mathbb{Q}_{[0,1]}$ defined s.t. for each $s \in X \cup Y$,

$$(\mathbb{X} \sqcup_k \mathbb{Y})(s) := \begin{cases} k \cdot \mathbb{X}(s) + (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in X \text{ and } s \in Y, \\ k \cdot \mathbb{X}(s) & \text{if } s \in X \text{ and } s \notin Y, \\ (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in Y \text{ and } s \notin X. \end{cases}$$

We then write that $Z = \mathbb{X} \sqcup \mathbb{Y}$ if $Z = \mathbb{X} \sqcup_k \mathbb{Y}$, for some k.

Intuition behind the disjunction

Question: How do we split distributions?
Answer: We rescale.

Let $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$ and $\mathbb{Y}\colon Y \to \mathbb{Q}_{[0,1]}$ be probabilistic teams and $k \in \mathbb{Q}_{[0,1]}$ be a rational number.

We denote by $\mathbb{X} \sqcup_k \mathbb{Y}$ the k-scaled union of $\mathbb{X}$ and $\mathbb{Y}$, that is, the probabilistic team $\mathbb{X} \sqcup_k \mathbb{Y}\colon X \cup Y \to \mathbb{Q}_{[0,1]}$ defined s.t. for each $s \in X \cup Y$,

$$(\mathbb{X} \sqcup_k \mathbb{Y})(s) := \begin{cases} k \cdot \mathbb{X}(s) + (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in X \text{ and } s \in Y, \\ k \cdot \mathbb{X}(s) & \text{if } s \in X \text{ and } s \notin Y, \\ (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in Y \text{ and } s \notin X. \end{cases}$$

We then write that $Z = \mathbb{X} \sqcup \mathbb{Y}$ if $Z = \mathbb{X} \sqcup_k \mathbb{Y}$, for some k.

Intuition behind the disjunction

Question: How do we split distributions?
Answer: We rescale.

Let $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$ and $\mathbb{Y}\colon Y \to \mathbb{Q}_{[0,1]}$ be probabilistic teams and $k \in \mathbb{Q}_{[0,1]}$ be a rational number.

We denote by $\mathbb{X} \sqcup_k \mathbb{Y}$ the k-scaled union of $\mathbb{X}$ and $\mathbb{Y}$, that is, the probabilistic team $\mathbb{X} \sqcup_k \mathbb{Y}\colon X \cup Y \to \mathbb{Q}_{[0,1]}$ defined s.t. for each $s \in X \cup Y$,

$$(\mathbb{X} \sqcup_k \mathbb{Y})(s) := \begin{cases} k \cdot \mathbb{X}(s) + (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in X \text{ and } s \in Y, \\ k \cdot \mathbb{X}(s) & \text{if } s \in X \text{ and } s \notin Y, \\ (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in Y \text{ and } s \notin X. \end{cases}$$

We then write that $Z = \mathbb{X} \sqcup \mathbb{Y}$ if $Z = \mathbb{X} \sqcup_k \mathbb{Y}$, for some k.

Example

Consider a database table that lists results of experiments as a multiteam or as the related probabilistic team using the counting measure.

- Records: Outcomes of measurements obtained simultaneously in two locations.
- Attributes: Test1 and Test2 ranging over types of measurements, and Outcome1 and Outcome2 ranging over outcomes of the measurements.

The probabilistic independence atom $\mathrm{Test1} \perp\!\!\!\perp \mathrm{Test2}$ expresses that the types of measurements are independently picked in the two locations.

The marginal identity atom $(\mathrm{Test1}, \mathrm{Outcome1}) \approx (\mathrm{Test2}, \mathrm{Outcome2})$ expresses that the distributions of tests and results are the same in both test sites.

The formula $\mathrm{Test1} = \mathrm{Test2} \vee (\mathrm{Test1} \neq \mathrm{Test2} \wedge \mathrm{Outcome1} \perp\!\!\!\perp \mathrm{Outcome2})$ expresses that there is no correlation between outcomes of the different measurements.

More examples

- The formula $\forall \vec{y}\, \vec{x} \approx \vec{y}$ states that the probabilities for $\overline{x}$ are uniformly distributed over all value sequences of length $|\overline{x}|$.
- The probability of $P(x)$ is at least twice the probability of $Q(x)$.
- Can we characterise the expressive power of $\mathrm{FO}(\approx, \perp\!\!\!\perp)$ in the probabilistic setting?

More examples

- The formula $\forall \vec{y}\, \vec{x} \approx \vec{y}$ states that the probabilities for $\overline{x}$ are uniformly distributed over all value sequences of length $|\overline{x}|$.
- The probability of $P(x)$ is at least twice the probability of $Q(x)$.
- Can we characterise the expressive power of $\mathrm{FO}(\approx, \perp\!\!\!\perp)$ in the probabilistic setting?

Benchmark logic

- In team semantics context fragments of second-order logic are captured.
- $\mathrm{FO}(\perp)$ (team semantics) is as expressive as existential second-order logic.
- We define a two-sorted variant of ESO in which we allow the quantification of rational distributions.
- This logic characterises the expressive power of $\mathrm{FO}(\approx, \perp\!\!\!\perp)$.

Benchmark logic

- In team semantics context fragments of second-order logic are captured.
- $\mathrm{FO}(\perp)$ (team semantics) is as expressive as existential second-order logic.
- We define a two-sorted variant of ESO in which we allow the quantification of rational distributions.
- This logic characterises the expressive power of $\mathrm{FO}(\approx, \perp\!\!\!\perp)$.

Probabilistic structures

Definition

Let τ and σ be a relational and a functional vocabulary. A probabilistic $\tau \cup \sigma$-structure is a tuple

$$\mathfrak{A} = (A, \mathbb{Q}_{[0,1]}, (R_i^{\mathfrak{A}})_{R_i \in \tau}, (f_i^{\mathfrak{A}})_{f_i \in \sigma}),$$

where

- A (i.e. the domain of $\mathfrak{A}$) is a finite nonempty set,
- $\mathbb{Q}_{[0,1]}$ is the set of rational numbers in the closed interval $[0, 1]$,
- each $R_i^{\mathfrak{A}}$ is a relation on A (i.e., a subset of $A^{\mathrm{ar}(R_i)}$),
- each $f_i^{\mathfrak{A}}$ is a probability distribution from $A^{\mathrm{ar}(f_i)}$ to $\mathbb{Q}_{[0,1]}$ (i.e., a function such that $\sum_{\vec{a} \in A^{\mathrm{ar}(f_i)}} f_i(\vec{a}) = 1$).

Second-order logic for probabilistic structures

- As first-order terms we have first-order variables.
- The set of numerical σ-terms i is defined via the grammar

$$i ::= f(\vec{x}) \mid i \times i \mid \mathrm{SUM}_{\vec{x}}\, i(\vec{x}, \vec{y}),$$

where $\vec{x}, \vec{y}$ are tuples of first-order variables, $f \in \sigma$ and σ is a set of functions.

- The value of a numerical term i in a structure $\mathfrak{A}$ under an assignment s is denoted by $[i]^{\mathfrak{A}}_s$ and defined as follows:

$$[f(\overline{x})]^{\mathfrak{A}}_s := f^{\mathfrak{A}}(s(\overline{x})), \qquad [i \times j]^{\mathfrak{A}}_s := [i]^{\mathfrak{A}}_s \cdot [j]^{\mathfrak{A}}_s,$$

$$[\mathrm{SUM}_{\vec{x}}\, i(\vec{x}, \vec{y})]^{\mathfrak{A}}_s := \sum_{\vec{a} \in A^{|\vec{x}|}} [i(\vec{a}, \vec{y})]^{\mathfrak{A}}_s,$$

where $\cdot$ and $\sum$ are the multiplication and sum of rational numbers.

Second-order logic for probabilistic structures

- As first-order terms we have first-order variables.
- The set of numerical σ-terms i is defined via the grammar

$$i ::= f(\vec{x}) \mid i \times i \mid \mathrm{SUM}_{\vec{x}}\, i(\vec{x}, \vec{y}),$$

  where $\vec{x}, \vec{y}$ are tuples of first-order variables, $f \in \sigma$ and σ is a set of functions.
- The value of a numerical term i in a structure $\mathfrak{A}$ under an assignment s is denoted by $[i]_s^{\mathfrak{A}}$ and defined as follows:

$$[f(\overline{x})]_s^{\mathfrak{A}} := f^{\mathfrak{A}}(s(\overline{x})), \qquad [i \times j]_s^{\mathfrak{A}} := [i]_s^{\mathfrak{A}} \cdot [j]_s^{\mathfrak{A}},$$

$$[\mathrm{SUM}_{\vec{x}}\, i(\vec{x}, \vec{y})]_s^{\mathfrak{A}} := \sum_{\vec{a} \in A^{|\vec{x}|}} [i(\vec{a}, \vec{y})]_s^{\mathfrak{A}},$$

  where $\cdot$ and $\sum$ are the multiplication and sum of rational numbers.

Second-order logic for probabilistic structures

Definition

The formulae of $\mathrm{ESOf}_{\mathbb{Q}}$ is defined via the following grammar:

$$\phi ::= x = y \mid x \neq y \mid i = j \mid i \neq j \mid R(\vec{x}) \mid \neg R(\vec{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi \mid \exists f \phi,$$

where i is a numerical term, R is a relation symbol, f is a function variable, $\vec{x}$ is a tuple of first-order variables.

Semantics of $\mathrm{ESOf}_{\mathbb{Q}}$ is defined via probabilistic structures and assignments analogous to FO. In addition to the clauses of first-order logic, we have:

$$\mathfrak{A} \models_s i = j \Leftrightarrow [i]^{\mathfrak{A}}_s = [j]^{\mathfrak{A}}_s, \qquad \mathfrak{A} \models_s i \neq j \Leftrightarrow [i]^{\mathfrak{A}}_s \neq [j]^{\mathfrak{A}}_s,$$

$$\mathfrak{A} \models_s \exists f \phi \Leftrightarrow \mathfrak{A}[h/f] \models_s \phi \text{ for some probability distribution } h \colon A^{\mathrm{ar}(f)} \to \mathbb{Q}_{[0,1]},$$

where $\mathfrak{A}[h/f]$ denotes the expansion of $\mathfrak{A}$ that interprets f to h.

Second-order logic for probabilistic structures

Definition

The formulae of $\mathrm{ESOf}_{\mathbb{Q}}$ is defined via the following grammar:

$$\phi ::= x = y \mid x \neq y \mid i = j \mid i \neq j \mid R(\vec{x}) \mid \neg R(\vec{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi \mid \exists f \phi,$$

where i is a numerical term, R is a relation symbol, f is a function variable, $\vec{x}$ is a tuple of first-order variables.

Semantics of $\mathrm{ESOf}_{\mathbb{Q}}$ is defined via probabilistic structures and assignments analogous to FO. In addition to the clauses of first-order logic, we have:

$$\mathfrak{A} \models_s i = j \Leftrightarrow [i]_s^{\mathfrak{A}} = [j]_s^{\mathfrak{A}}, \qquad \mathfrak{A} \models_s i \neq j \Leftrightarrow [i]_s^{\mathfrak{A}} \neq [j]_s^{\mathfrak{A}},$$

$$\mathfrak{A} \models_s \exists f \phi \Leftrightarrow \mathfrak{A}[h/f] \models_s \phi \text{ for some probability distribution } h\colon A^{\mathrm{ar}(f)} \to \mathbb{Q}_{[0,1]},$$

where $\mathfrak{A}[h/f]$ denotes the expansion of $\mathfrak{A}$ that interprets f to h.

Examples

- Uniformity of a distribution f can be expressed with

$$\phi(f) := \forall \overline{xy}(f(\overline{x}) = 0 \vee f(\overline{y}) = 0 \vee f(\overline{x}) = f(\overline{y})).$$

- For a numerical term i and rational number $\frac{p}{q}$, the property

$$i(\overline{x}) = \frac{p}{q}$$

can be expressed in $\mathrm{ESOf}_{\mathbb{Q}}$.

Translating from $\mathrm{FO}(\perp\!\!\!\perp, \approx)$ to $\mathrm{ESOf}_{\mathbb{Q}}$

For a probabilistic team $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$, we let $f_{\mathbb{X}} : A^n \to \mathbb{Q}_{[0,1]}$ be the probability distribution such that $f_{\mathbb{X}}(s(\overline{x})) = \mathbb{X}(s)$ for all $s \in X$.

Theorem

For every $\phi(\overline{x}) \in \mathrm{FO}(\perp\!\!\!\perp, \approx)$ there is a formula $\phi^*(f) \in \mathrm{ESOf}_{\mathbb{Q}}$ with one free function variable f s.t. for all structures $\mathfrak{A}$ and nonempty probabilistic teams $\mathbb{X}$

$$\mathfrak{A} \models_{\mathbb{X}} \phi(\overline{x}) \iff (\mathfrak{A}, f_{\mathbb{X}}) \models \phi^*(f).$$

Translating from $\mathrm{FO}(\perp\!\!\!\perp, \approx)$ to $\mathrm{ESOf}_{\mathbb{Q}}$

For a probabilistic team $\mathbb{X}\colon X \to \mathbb{Q}_{[0,1]}$, we let $f_{\mathbb{X}} : A^n \to \mathbb{Q}_{[0,1]}$ be the probability distribution such that $f_{\mathbb{X}}(s(\overline{x})) = \mathbb{X}(s)$ for all $s \in X$.

Theorem

For every $\phi(\overline{x}) \in \mathrm{FO}(\perp\!\!\!\perp, \approx)$ there is a formula $\phi^*(f) \in \mathrm{ESOf}_{\mathbb{Q}}$ with one free function variable f s.t. for all structures $\mathfrak{A}$ and nonempty probabilistic teams $\mathbb{X}$

$$\mathfrak{A} \models_{\mathbb{X}} \phi(\overline{x}) \iff (\mathfrak{A}, f_{\mathbb{X}}) \models \phi^*(f).$$

Translating from $\mathrm{ESOf}_{\mathbb{Q}}$ to $\mathrm{FO}(\perp\!\!\!\perp, \approx)$

- The translation is more involved.
- The proof utilises the observation that independence atoms and marginal identity atoms can be used to express multiplication and SUM in $\mathbb{Q}_{[0,1]}$.

Lemma

Every $\mathrm{ESOf}_{\mathbb{Q}}$ sentence is equivalent to a sentence of the form $\exists\overline{f}\forall\overline{x}\theta$, where θ is quantifier-free and such that its second sort identity atoms are of the form $f_i(\overline{uv}) = f_j(\overline{u}) \times f_k(\overline{v})$ or $f_i(\overline{u}) = \mathrm{SUM}_{\overline{v}} f_j(\overline{uv})$ for distinct f_i, f_j, f_k.

Theorem

Let $\phi(p) \in \mathrm{ESOf}_{\mathbb{Q}}$ be a sentence with exactly one free function symbol p in the normal form of the lemma above. Then there is a formula $\Phi \in \mathrm{FO}(\perp\!\!\!\perp, \approx)$ such that for all structures $\mathfrak{A}$ and probabilistic teams $\mathbb{X} := p^{\mathfrak{A}}$,

$$\mathfrak{A} \models_{\mathbb{X}} \Phi \iff (\mathfrak{A}, p) \models \phi.$$

Translating from $\mathrm{ESOf}_{\mathbb{Q}}$ to $\mathrm{FO}(\perp\!\!\!\perp, \approx)$

- The translation is more involved.
- The proof utilises the observation that independence atoms and marginal identity atoms can be used to express multiplication and SUM in $\mathbb{Q}_{[0,1]}$.

Lemma

Every $\mathrm{ESOf}_{\mathbb{Q}}$ sentence is equivalent to a sentence of the form $\exists \overline{f} \forall \overline{x} \theta$, where θ is quantifier-free and such that its second sort identity atoms are of the form $f_i(\overline{uv}) = f_j(\overline{u}) \times f_k(\overline{v})$ or $f_i(\overline{u}) = \mathrm{SUM}_{\overline{v}} f_j(\overline{uv})$ for distinct f_i, f_j, f_k.

Theorem

Let $\phi(p) \in \mathrm{ESOf}_{\mathbb{Q}}$ be a sentence with exactly one free function symbol p in the normal form of the lemma above. Then there is a formula $\Phi \in \mathrm{FO}(\perp\!\!\!\perp, \approx)$ such that for all structures $\mathfrak{A}$ and probabilistic teams $\mathbb{X} := p^{\mathfrak{A}}$,

$$\mathfrak{A} \models_{\mathbb{X}} \Phi \iff (\mathfrak{A}, p) \models \phi.$$

Translating from $\mathrm{ESOf}_{\mathbb{Q}}$ to $\mathrm{FO}(\perp\!\!\!\perp, \approx)$

Lemma

Every $\mathrm{ESOf}_{\mathbb{Q}}$ *sentence is equivalent to a sentence of the form* $\exists \overline{f} \forall \overline{x} \theta$, *where* θ *is quantifier-free and such that its second sort identity atoms are of the form* $f_i(\overline{uv}) = f_j(\overline{u}) \times f_k(\overline{v})$ *or* $f_i(\overline{u}) = \mathrm{SUM}_{\overline{v}} f_j(\overline{uv})$ *for distinct* f_i, f_j, f_k.

Theorem

Let $\phi(p) \in \mathrm{ESOf}_{\mathbb{Q}}$ *be a sentence with exactly one free function symbol* p *in the normal form of the lemma above. Then there is a formula* $\Phi \in \mathrm{FO}(\perp\!\!\!\perp, \approx)$ *such that for all structures* $\mathfrak{A}$ *and probabilistic teams* $\mathbb{X} := p^{\mathfrak{A}}$,

$$\mathfrak{A} \models_{\mathbb{X}} \Phi \iff (\mathfrak{A}, p) \models \phi.$$

Descriptive complexity

- FO(⊥) (team semantics) is equi-expressive to ESO and thus captures NP.
- FO(⊆) (team semantics) is equi-expressive to positive greatest fixed point-logic and thus captures P on ordered structures.
- FO(≈) (multiteam and probabilistic team seamantics) is the probabilistic or counting variant of FO(⊆). It is thus interesting to see how complex problems can be expressed in it.
- In multiteam setting FO(≈) can express NP-complete problems:
  Exact cover problem:
  Input: A collection $\mathcal{S}$ of subsets of a set A.
  Decide: Does there exist a subcollection $\mathcal{S}^*$ of $\mathcal{S}$ such that each element in A is contained in exactly one subset in $\mathcal{S}^*$?

Descriptive complexity

- $\mathrm{FO}(\perp)$ (team semantics) is equi-expressive to ESO and thus captures NP.
- $\mathrm{FO}(\subseteq)$ (team semantics) is equi-expressive to positive greatest fixed point-logic and thus captures P on ordered structures.
- $\mathrm{FO}(\approx)$ (multiteam and probabilistic team seamantics) is the probabilistic or counting variant of $\mathrm{FO}(\subseteq)$. It is thus interesting to see how complex problems can be expressed in it.
- In multiteam setting $\mathrm{FO}(\approx)$ can express NP-complete problems:
  Exact cover problem:
  Input: A collection $\mathcal{S}$ of subsets of a set A.
  Decide: Does there exist a subcollection $\mathcal{S}^*$ of $\mathcal{S}$ such that each element in A is contained in exactly one subset in $\mathcal{S}^*$?

Descriptive complexity

- FO(⊥) (team semantics) is equi-expressive to ESO and thus captures NP.
- FO(⊆) (team semantics) is equi-expressive to positive greatest fixed point-logic and thus captures P on ordered structures.
- FO(≈) (multiteam and probabilistic team seamantics) is the probabilistic or counting variant of FO(⊆). It is thus interesting to see how complex problems can be expressed in it.
- In multiteam setting FO(≈) can express NP-complete problems:
  Exact cover problem:
  Input: A collection $\mathcal{S}$ of subsets of a set A.
  Decide: Does there exist a subcollection $\mathcal{S}^*$ of $\mathcal{S}$ such that each element in A is contained in exactly one subset in $\mathcal{S}^*$?

Exact cover example

Multiteam $\mathcal{X}$

element	set	left	right	$\mathcal{X}(s)$
0	S_1	1	2	1
0	S_1	2	3	1
0	S_1	3	1	1
0	S_2	2	2	1
0	S_3	1	3	1
0	S_3	3	4	1
0	S_3	4	1	1
1	0	0	0	1
2	0	0	0	1
3	0	0	0	1
4	0	0	0	1

Consider an exact cover problem over $A = \{1,2,3,4\}$ and $\mathcal{S} = \{S_1 = \{1,2,3\}, S_2 = \{2\}, S_3 = \{1,3,4\}\}$. Our constructed multiteam $\mathcal{X}$ is depicted on left.

The answer to the exact cover problem is positive iff $\mathcal{X}$ satisfies the formula

$$\mathtt{set} \neq 0 \vee$$

$$(\mathtt{element} \approx \mathtt{left} \wedge (\mathtt{set}, \mathtt{right}) \approx (\mathtt{set}, \mathtt{left}))$$

Theorem

Data complexity of $\mathrm{FO}(\approx)$ and the quantifier-free fragment of $\mathrm{FO}(\approx)$ under multiteam semantics are NP-complete.

Exact cover example

Multiteam $\mathcal{X}$

element	set	left	right	$\mathcal{X}(s)$
0	S_1	1	2	1
0	S_1	2	3	1
0	S_1	3	1	1
0	S_2	2	2	1
0	S_3	1	3	1
0	S_3	3	4	1
0	S_3	4	1	1
1	0	0	0	1
2	0	0	0	1
3	0	0	0	1
4	0	0	0	1

Consider an exact cover problem over $A = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{S_1 = \{1, 2, 3\}, S_2 = \{2\}, S_3 = \{1, 3, 4\}\}$. Our constructed multiteam $\mathcal{X}$ is depicted on left.

The answer to the exact cover problem is positive iff $\mathcal{X}$ satisfies the formula

$$\mathtt{set} \neq 0 \vee$$
$$\big(\mathtt{element} \approx \mathtt{left} \wedge (\mathtt{set}, \mathtt{right}) \approx (\mathtt{set}, \mathtt{left})\big)$$

Theorem

Data complexity of $\mathrm{FO}(\approx)$ *and the quantifier-free fragment of* $\mathrm{FO}(\approx)$ *under multiteam semantics are* NP*-complete.*

Exact cover example

Multiteam $\mathcal{X}$

element	set	left	right	$\mathcal{X}(s)$
0	S_1	1	2	1
0	S_1	2	3	1
0	S_1	3	1	1
0	S_2	2	2	1
0	S_3	1	3	1
0	S_3	3	4	1
0	S_3	4	1	1
1	0	0	0	1
2	0	0	0	1
3	0	0	0	1
4	0	0	0	1

Consider an exact cover problem over $A = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{S_1 = \{1, 2, 3\}, S_2 = \{2\}, S_3 = \{1, 3, 4\}\}$. Our constructed multiteam $\mathcal{X}$ is depicted on left.

The answer to the exact cover problem is positive iff $\mathcal{X}$ satisfies the formula

$$\texttt{set} \neq 0 \vee$$
$$(\texttt{element} \approx \texttt{left} \wedge (\texttt{set}, \texttt{right}) \approx (\texttt{set}, \texttt{left}))$$

Theorem

Data complexity of $\mathrm{FO}(\approx)$ and the quantifier-free fragment of $\mathrm{FO}(\approx)$ under multiteam semantics are NP*-complete.*

Exact cover example

Thanks!

Multiteam $\mathcal{X}$

element	set	left	right	$\mathcal{X}(s)$
0	S_1	1	2	1
0	S_1	2	3	1
0	S_1	3	1	1
0	S_2	2	2	1
0	S_3	1	3	1
0	S_3	3	4	1
0	S_3	4	1	1
1	0	0	0	1
2	0	0	0	1
3	0	0	0	1
4	0	0	0	1

Consider an exact cover problem over $A = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{S_1 = \{1, 2, 3\}, S_2 = \{2\}, S_3 = \{1, 3, 4\}\}$. Our constructed multiteam $\mathcal{X}$ is depicted on left.

The answer to the exact cover problem is positive iff $\mathcal{X}$ satisfies the formula

$$\mathtt{set} \neq 0 \vee$$
$$(\mathtt{element} \approx \mathtt{left} \wedge (\mathtt{set}, \mathtt{right}) \approx (\mathtt{set}, \mathtt{left}))$$

Theorem

Data complexity of $\mathrm{FO}(\approx)$ *and the quantifier-free fragment of* $\mathrm{FO}(\approx)$ *under multiteam semantics are* NP*-complete.*