# On the Complexity of Horn and Krom Fragments of Second-Order Boolean Logic

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# Canonical complete problems

Quantified Boolean formula

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists p \mid \forall p$$

| SAT (Cook, 1971) | |
|---|---|
| Input: | Boolean formula $\theta$ |
| Question: | Is $\theta$ satisfiable? |
| Complete for: | NP |

| QBF (Stockmeyer and Meyer, 1973) | |
|---|---|
| Input: | Quantified Boolean formula $\varphi := Q_1 p_1 \dots Q_n p_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PSPACE |

W.l.o.g. $\theta$ in 3CNF

$$\theta = (p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_4 \vee p_5) \wedge \dots$$

# Complete problems from propositional logic

| SAT (Cook, 1971) | |
|---|---|
| Input: | Boolean formula $\theta$ |
| Question: | Is $\theta$ satisfiable? |
| Complete for: | NP |

| QBF (Stockmeyer and Meyer, 1973) | |
|---|---|
| Input: | Quantified Boolean formula $\varphi := \forall p_1 \ldots \forall p_m \exists q_1 \ldots \exists q_n \theta$ and constraints $\vec{c}_1, \ldots, \vec{c}_n$, $c_i \subseteq c_{i+1}$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PSPACE |

| DQBF (Peterson, Reif, Azhar, 2001) | |
|---|---|
| Input: | Dependency Quantified Boolean formula $\varphi := \forall p_1 \ldots \forall p_m \exists q_1 \ldots \exists q_n \theta$ and constraints $\vec{c}_1, \ldots, \vec{c}_n$ |
| Question: | Is $\varphi$ true? |
| Complete for: | NEXP |

W.l.o.g. $\theta$ in 3CNF

$$\theta = (p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_4 \vee p_5) \wedge \ldots$$

# Complete problems from propositional logic

| HORNSAT (Dowling, Gallier, 1984) | |
|---|---|
| Input: | Horn formula $\theta$ |
| Question: | Is $\theta$ satisfiable? |
| Complete for: | PTIME |

| QHORN (Karpinski, Kleine Büning, Schmitt, CSL 1987) | |
|---|---|
| Input: | Quantified Horn formula $Q_1 p_1 \dots Q_n p_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

| DQHORN (Bubeck, Kleine Büning, 2006) | |
|---|---|
| Input: | Dependency Quantified Horn formula $\varphi := \forall p_1 \dots \forall p_m \exists q_1 \dots \exists q_n \theta$ and constraints $\vec{c}_1, \dots, \vec{c}_n$ |
| Question: | Is $\varphi$ true? |
| Complete for: | ? |

$\theta$ in Horn form

$$\theta = (p_1 \wedge p_2 \rightarrow p_3) \wedge (p_2 \wedge p_4 \wedge p_7 \rightarrow p_5) \wedge \dots$$

# Complete problems from propositional logic

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| Input: | Horn formula $\theta$ |
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| Input: | Quantified Horn formula $Q_1 p_1 \dots Q_n p_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

| DQHORN (Bubeck, Kleine Büning, 2006) | |
|---|---|
| Input: | Dependency Quantified Horn formula $\varphi := \forall p_1 \dots \forall p_m \exists q_1 \dots \exists q_n \theta$ and constraints $\vec{c}_1, \dots, \vec{c}_n$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

$\theta$ in Horn form

$$\theta = (p_1 \wedge p_2 \rightarrow p_3) \wedge (p_2 \wedge p_4 \wedge p_7 \rightarrow p_5) \wedge \dots$$

# Complete problems from propositional logic

**HORNSAT** (Dowling, Gallier, 1984)

| | |
|---|---|
| Input: | Existential first-order Boolean Horn formula $\varphi := \exists x_1 \dots \exists x_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

**QHORN** (Karpinski, Kleine Büning, Schmitt, CSL 1987)

| | |
|---|---|
| Input: | Quantified Horn formula $Q_1 p_1 \dots Q_n p_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

**DQHORN** (Bubeck, Kleine Büning, 2006)

| | |
|---|---|
| Input: | Existential second-order Horn formula $\varphi := \exists f_1 \dots \exists f_n \forall p_1 \dots \forall p_m \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

$\theta$ in Horn form

$$\theta = (p_1 \wedge p_2 \rightarrow p_3) \wedge (p_2 \wedge p_4 \wedge p_7 \rightarrow p_5) \wedge \dots$$

# Complete problems from propositional logic

| HORNSAT (Dowling, Gallier, 1984) | |
|---|---|
| Input: | Existential first-order Boolean Horn formula $\varphi := \exists x_1 \ldots \exists x_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

| QHORN (Karpinski, Kleine Büning, Schmitt, CSL 1987) | |
|---|---|
| Input: | Quantified Horn formula $Q_1 p_1 \ldots Q_n p_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

| DQHORN (Bubeck, Kleine Büning, 2006) | |
|---|---|
| Input: | Existential second-order Horn formula $\varphi := \exists f_1 \ldots \exists f_n \forall p_1 \ldots \forall p_m \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

$\theta$ in Horn form

$$\theta = (p_1 \wedge p_2 \rightarrow p_3) \wedge (p_2 \wedge p_4 \wedge p_7 \rightarrow p_5) \wedge \ldots$$

More expressive second-order quantification?

# Problem setting

Start with a CNF formula $\varphi$.

(1) Restrict structure (from CNF to Horn )

$\Longrightarrow$ reduce complexity: $\{\text{NP}, \text{PSPACE}, \text{NEXPTIME}\} \Rightarrow \text{PTIME}$

(2) Add quantification (from existential first-order to existential second-order )

$\Longrightarrow$ increase complexity (CNF): $\text{NP} \Rightarrow \text{NEXPTIME}$

$\Longrightarrow$ no increase complexity (HORN): $\text{PTIME} \Rightarrow \text{PTIME}$

# Problem setting

Start with a CNF formula $\varphi$.

(1) Restrict structure (from CNF to ________________?)
$\Longrightarrow$ reduce complexity: ?

(2) Add quantification (from existential first-order to ________________?)
$\Longrightarrow$ increase complexity ?

# Problem setting

Start with a CNF formula $\varphi$.

(1) Restrict structure (from CNF to ______________?)
$\Longrightarrow$ reduce complexity: ?

(2) Add quantification (from existential first-order to ______________?)
$\Longrightarrow$ increase complexity ?

**Metalevel problem:**

Computational complexity of Boolean formula $\varphi$ with

- complex (second-order) quantification
- simple quantifier-free structure

# Second-order Boolean logic – Syntax

- functions $f$ of arity $0$ are *propositional*; otherwise *proper function variables*
- terms consist of:
  - propositional variables $f$,
  - expressions $f(t_1, \ldots, t_n)$, where $f$ is a proper function variable and $t_i$ are terms

## Definition (Second-order Boolean logic)

Second-order Boolean logic ($\mathrm{SO}_2$) consists of formulae:

$$\varphi ::= t \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists f \varphi,$$

where $t$ is a term.

# Second-order Boolean logic – Semantics

- *Interpretation* $I$ maps every variable $f$ to $I(f)\colon \{0,1\}^{\mathrm{ar}(f)} \to \{0,1\}$
- *Valuation* $[\![\varphi]\!]_I \in \{0,1\}$ of a formula $\varphi$ defined as:

$$
\begin{aligned}
[\![\varphi \wedge \psi]\!]_I &:= [\![\varphi]\!]_I \cdot [\![\psi]\!]_I, \\
[\![\neg\varphi]\!]_I &:= 1 - [\![\varphi]\!]_I, \\
[\![f(\varphi_1, \ldots, \varphi_n)]\!]_I &:= I(f)([\![\varphi_1]\!]_I, \ldots, [\![\varphi_n]\!]_I), \\
[\![\exists f \varphi]\!]_I &:= \max\{[\![\varphi]\!]_{I_F^f} | F : \{0,1\}^n \to \{0,1\}\},
\end{aligned}
$$

where $I_F^f$ obtained from $I$ by replacing $I(f)$ with $F$.

# Second-order Boolean logic – Complexity I

| Truth problem | |
|---|---|
| Input: | Closed $SO_2$-formula $\varphi$ |
| Question: | Is $\varphi$ true? |

**Theorem (Lohrey 2012)**

*Truth of $SO_2$-formulae is complete for alternating exponential time with polynomially many alternations ($\mathsf{AEXP}(\mathrm{poly})$)*

# DQHORN as a fragment of $SO_2$

DQHORN instance:

$$\forall p_1 \dots \forall p_m \exists q_1 \dots \exists q_n \theta \text{ and constraints } \vec{c}_i \subseteq \{p_1, \dots, p_m\}^1$$

$$\equiv \quad \exists f_1 \dots \exists f_n \forall p_1 \dots \forall p_m \theta(f_1(\vec{c}_1)/q_1, \dots, f_n(\vec{c}_n)/q_n) \in SO_2$$

Observations:

1) No (second-order) quantifier alternation

2) Simple clause structure (Horn)

3) Simple term structure
    - no nested function terms
    - unique function arguments

[1] $q_i$ may depend only on $\vec{c}_i$

# DQHORN as a fragment of $SO_2$

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Observations:

1) No (second-order) quantifier alternation
2) Simple clause structure (Horn)
3) Simple term structure
   - no nested function terms
   - unique function arguments

[1] $q_i$ may depend only on $\vec{c}_i$

# 1) Second-order quantifier alternation

## Definition

$\Sigma_k$ consists of all $\mathsf{SO}_2$ formulae of the form

$$Q_1 \vec{f_1} \cdots Q_k \vec{f_k}\ Q_{k+1}\, \vec{x}\, \theta,$$

where $Q_i = \exists$ ($Q_i = \forall$) if $i$ is odd (even), $\theta$ is quantifier-free, and $\vec{x}$ is a tuple of propositional variables. For $\Pi_k$ swap $\exists$ and $\forall$.

## Theorem ([Lohrey, 2012])

*Truth of $\Sigma_k$-formulae is complete for $\Sigma_k^{\mathsf{E}}$, and truth of $\Pi_k$-formulae is complete for $\Pi_k^{\mathsf{E}}$.*

Levels of the exponential time hierarchy: $\Sigma_k^{\mathsf{E}} := \mathsf{NEXP}^{\Sigma_{k-1}^{\mathsf{P}}}$ and $\Pi_k^{\mathsf{E}} := \mathsf{coNEXP}^{\Sigma_{k-1}^{\mathsf{P}}}$

# 2) Clause structure

| HORNSAT | |
|---|---|
| Input: | Horn formula $\theta$ |
| Question: | Is $\theta$ satisfiable? |
| Complete for: | PTIME |

| QHORN | |
|---|---|
| Input: | Quantified Horn formula $\varphi := Q_1 x_1 \dots Q_n x_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

| DQHORN | |
|---|---|
| Input: | Quantified Horn formula $\varphi := \forall p_1 \dots \forall p_m \exists q_1 \dots \exists q_n \theta$ and constraints $C_1, \dots, C_n$ |
| Question: | Is $\varphi$ true? |
| Complete for: | PTIME |

$\theta$ in Horn form

$$\theta = (p_1 \wedge p_2 \to p_3) \wedge (p_2 \wedge p_4 \wedge p_7 \to p_5) \wedge \dots$$

# 2) Clause structure

| 2SAT (Jones et al. 76, Immerman 88, Szelepcsényi 88) | |
|---|---|
| Input: | Krom formula $\theta$ |
| Question: | Is $\theta$ satisfiable? |
| Complete for: | NL |

| QKROM (Aspvall, Plass, Tarjan, 1979) | |
|---|---|
| Input: | Quantified Krom formula $\varphi := Q_1 x_1 \dots Q_n x_n \theta$ |
| Question: | Is $\varphi$ true? |
| Complete for: | NL |

| DQKROM | |
|---|---|
| Input: | Quantified Krom formula $\varphi := \forall p_1 \dots \forall p_m \exists q_1 \dots \exists q_n \theta$ and constraints $C_1, \dots, C_n$ |
| Question: | Is $\varphi$ true? |
| Complete for: | ? |

$\theta$ in Krom (2CNF) form

$$\theta = (p_1 \vee \neg p_3) \wedge (\neg p_3 \vee \neg p_4) \wedge \dots$$

# 2) Clause structure

CNF formula $(C_1 \land \ldots \land C_n) \in \mathrm{SO}_2$ called

a) *Horn* if each clause $C_i$ contains at most one positive literal.
$\neg p \lor \neg f(g(q)) \lor f(p,q)$ or written as an implication $p \land f(g(q)) \rightarrow f(p,q)$

b) *Krom* if each clause $C_i$ contains at most two literals.
$\neg f(p) \lor \neg g(p,q)$

c) *core* if it is both Horn and Krom.
$\neg f(p,q,r) \lor g(h(p))$ or written as an implication $f(p,q,r) \rightarrow g(h(p))$

# 3) Term structure

Example:
In DQHORN quantifier-free part translates to $\theta(f_1(\vec{c}_1)/p_i, \ldots, f_n(\vec{c}_n)/p_n)$

A formula $\varphi \in \mathsf{SO}_2$ is

a) *unique*: $f(t_1, \ldots, t_n)$ and $f(t'_1, \ldots, t'_n)$ occur in $\varphi$
$\implies t_i = t'_i$ for all $i \in [n]$.

b) *simple*: $f(t_1, \ldots, t_n)$ occurs in $\varphi$
$\implies t_i$ are propositional variables.

# Fragments of $SO_2$ – Examples

| formula | fragment | Horn | Krom | core | unique | simple |
|---|---|---|---|---|---|---|
| $\exists fg \forall pqr \big(\neg f(q) \vee p \vee \neg g(p,q)\big) \wedge \big(f(q) \vee \neg f(r)\big)$ | $\Sigma_1$ | ✓ | | | | ✓ |
| $\exists f \forall g \exists pq \big(f(g(p)) \vee p\big) \wedge \big(\neg q \vee \neg f(g(p))\big)$ | $\Sigma_2$ | | ✓ | | ✓ | |
| $\forall f \exists gh \forall pq \big(f(p) \vee \neg g(q)\big) \wedge \big(\neg h(p,q) \vee \neg f(p)\big)$ | $\Pi_2$ | ✓ | ✓ | ✓ | ✓ | ✓ |

# Fragments of $SO_2$ – Examples

| formula | fragment | Horn | Krom | core | unique | simple |
|---|---|---|---|---|---|---|
| $\exists fg \forall pqr \Big(\neg f(q) \vee p \vee \neg g(p,q)\Big) \wedge \Big(f(q) \vee \neg f(r)\Big)$ | $\Sigma_1$ | ✓ | | | | ✓ |
| $\exists f \forall g \exists pq \Big(f(g(p)) \vee p\Big) \wedge \Big(\neg q \vee \neg f(g(p))\Big)$ | $\Sigma_2$ | | ✓ | | ✓ | |
| $\forall f \exists gh \forall pq \Big(f(p) \vee \neg g(q)\Big) \wedge \Big(\neg h(p,q) \vee \neg f(p)\Big)$ | $\Pi_2$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| DQHORN | $\Sigma_1$ | ✓ | | | ✓ | ✓ |

# Summary

| Simpleness | Uniqueness | $k$ | Clauses | $\Sigma_k$ | $\Pi_k$ | Reference |
|---|---|---|---|---|---|---|
| Simple | Unique | $k = 1$ | Horn | P | | [Bubeck and Kleine Büning, 2006] |
| | | | Krom/core | | | |
| | | $k = 2$ | Horn<br>Krom/core | | | |
| | | $k \geq 3$ odd | $\star$ | | | |
| | | $k \geq 4$ even | $\star$ | | | |
| | Non-unique | $k = 1$ | Horn | | | |
| | | | Krom/core | | | |
| | | $k \geq 3$ odd | $\star$ | | | |
| | | $k \geq 2$ even | $\star$ | | | |
| Non-simple | Unique | $k = 1$ | Horn | | | |
| | | | Krom/core | | | |
| | | $k \geq 2$ | $\star$ | | | |
| | Non-unique | $k \geq 1$ | $\star$ | | | |
| $\star$ | $\star$ | $k = \omega$ | $\star$ | AEXP(poly) | AEXP(poly) | H: t, [Hannula et al., 2016], $\in$: [Lohrey, 2012] |

$\star$: "any", "H": hardness, "$\in$": membership, "t": this paper, †: likely identical with first row. ‡: follows from some other result in the table.

# Unique and simple Krom $\Sigma_1$ I

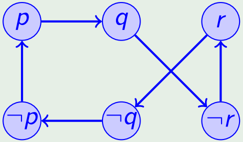
## Definition (Implication graph)

A quantifier-free Krom formula $\theta$ gives rise to *implication graph* $G = (V, E)$, where

- $V$ contains all literals $l$ in $\varphi$ (closed under $\neg$, $\neg\neg l$ identified with $l$)
- $E$ contains an edge $l_1 \to l_2$ for each clause $\neg l_1 \vee l_2$ in $\varphi$, and an edge $\neg l \to l$ for each unit clause $l$ in $\varphi$.

## Example

$(\neg p \vee q) \wedge (\neg q \vee \neg r) \wedge p \wedge r$ corresponds to



# Unique and simple Krom $\Sigma_1$ II

SAT over Krom (2CNF) formulae is true iff the following conditions all hold [Aspvall et al., 1979]:

(1) No vertices $v$ and $\neg v$ are in the same scc.

**scc**: strongly connected component

# Unique and simple Krom $\Sigma_1$ III

QBF over Krom (2CNF) formulae is true iff the following conditions all hold [Aspvall et al., 1979]:

(1) There is no path from a universal vertex $u$ to another universal vertex $u'$ (with $u \neq u'$, but possibly $u = \neg u'$).

(2) No vertices $v$ and $\neg v$ are in the same scc.

(3) Every existential vertex $v$ in the same scc as a universal vertex $u$ must depend on $u$.

**scc**: strongly connected component

# Unique and simple Krom $\Sigma_1$ IV

We show that unique and simple $\Sigma_1$ over Krom (2CNF) formulae is true iff the following conditions all hold:

(1) There is no path from a universal vertex $u$ to another universal vertex $u'$ (with $u \neq u'$, but possibly $u = \neg u'$).

(2) No vertices $v$ and $\neg v$ are in the same scc.

(3) Every existential vertex $v$ in the same scc as a universal vertex $u$ must depend on $u$.

(4) There is no $\rightsquigarrow$-cycle among the scc's (including loops).

$v$ *depends on* $v'$, written $v \rightsquigarrow v'$, if e.g. $v'$ is an argument of $v$.

If $S$ and $S'$ are scc's, then $S \rightsquigarrow S'$ if some *universal* vertex $u \in S$ depends on some vertex $v \in S'$ (with possibly $S = S'$)

# Unique and simple Krom $\Sigma_1$ V

We show that unique and simple $\Sigma_1$ over Krom (2CNF) formulae is true iff the following conditions all hold:

(1) There is no path from a universal vertex $u$ to another universal vertex $u'$ (with $u \neq u'$, but possibly $u = \neg u'$).

(2) No vertices $v$ and $\neg v$ are in the same scc.

(3) Every existential vertex $v$ in the same scc as a universal vertex $u$ must depend on $u$.

(4) There is no $\rightsquigarrow$-cycle among the scc's (including loops).

**Theorem**

*Truth of unique and simple Krom $\Sigma_1$ is NL-complete.*

# Unique and simple Krom $\Sigma_1$ VI

We show that unique and simple $\Sigma_1$ over Krom (2CNF) formulae is true iff the following conditions all hold:

(1) There is no path from a universal vertex $u$ to another universal vertex $u'$ (with $u \neq u'$, but possibly $u = \neg u'$).

(2) No vertices $v$ and $\neg v$ are in the same scc.

(3) Every existential vertex $v$ in the same scc as a universal vertex $u$ must depend on $u$.

(4) There is no $\rightsquigarrow$-cycle among the scc's (including loops).

## Example

$\forall f_1 \forall f_2 \exists x_1 \exists x_2 (f_1(x_2) \leftrightarrow x_1) \wedge (f_2(x_1) \leftrightarrow x_2)$ violates (4). Here $f_1(x_2) \rightsquigarrow x_2$ and $f_2(x_1) \rightsquigarrow x_1$, and therefore $\{f_1(x_2), x_1\} \rightsquigarrow \{x_2, f_2(x_1)\} \rightsquigarrow \{f_1(x_2), x_1\}$ on the level of components.
Indeed, choosing the universal quantifiers as $f_1(x_2) = \neg x_2, f_2(x_1) = x_1$ refutes the formula.

# Summary

| Simpleness | Uniqueness | $k$ | Clauses | $\Sigma_k$ | $\Pi_k$ | Reference |
|---|---|---|---|---|---|---|
| Simple | Unique | $k = 1$ | Horn | P | | [Bubeck and Kleine Büning, 2006] |
| | | | Krom/core | NL | NL | H/$\in$: t H/$\in$: t |
| | | $k = 2$ | Horn | | | |
| | | | Krom/core | | NL | H/$\in$: t |
| | | $k \geq 3$ odd | $\star$ | | | |
| | | $k \geq 4$ even | $\star$ | | | |
| | Non-unique | $k = 1$ | Horn | | | |
| | | | Krom/core | | | |
| | | $k \geq 3$ odd | $\star$ | | | |
| | | $k \geq 2$ even | $\star$ | | | |
| Non-simple | Unique | $k = 1$ | Horn | | | |
| | | | Krom/core | | NL | H/$\in$: t |
| | | $k \geq 2$ | $\star$ | | | |
| | Non-unique | $k \geq 1$ | $\star$ | | | |
| $\star$ | $\star$ | $k = \omega$ | $\star$ | AEXP(poly) | AEXP(poly) | H: t, [Hannula et al., 2016], $\in$: [Lohrey, 2012] |

$\star$: "any", "H": hardness, "$\in$": membership, "t": this paper, †: likely identical with first row. ‡: follows from some other result in the table.

# Without uniqueness/simpleness

**Theorem**

*Truth of unique Horn/Krom/core $\Sigma_k$ is $\Sigma_k^E$-complete.*

**Theorem**

*Truth of simple Krom/core $\Sigma_1$ is PSPACE-complete.*
*Truth of simple Horn $\Sigma_1$ is EXP-complete.*

Otherwise, simple Krom/core/Horn $\Sigma_k$ and $\Pi_k$ correspond to levels of the exponential time hierarchy.

# Summary

| Simpleness | Uniqueness | $k$ | Clauses | $\Sigma_k$ | $\Pi_k$ | Reference | |
|---|---|---|---|---|---|---|---|
| Simple | Unique | $k = 1$ | Horn | P | ? | [Bubeck and Kleine Büning, 2006] | |
| | | | Krom/core | NL | NL | H/$\in$: t | H/$\in$: t |
| | | $k = 2$ | Horn | $\Sigma_2^E$ | ? | H/$\in$: ‡ | |
| | | | Krom/core | $\Sigma_2^E$ | NL | H: t, $\in$: ‡ | H/$\in$: t |
| | | $k \geq 3$ odd | $\star$ | $\Sigma_{k-1}^E$ | $\Pi_k^E$ | H: t $\in$: ‡ | H: t $\in$: ‡ |
| | | $k \geq 4$ even | $\star$ | $\Sigma_k^E$ | $\Pi_{k-1}^E$ | H: t $\in$: ‡ | H: t $\in$: ‡ |
| | Non-unique | $k = 1$ | Horn | EXP | $\Pi_1^E$ | H/$\in$: t | H/$\in$: ‡ |
| | | | Krom/core | PSPACE | $\Pi_1^E$ | H/$\in$: t | H: t, $\in$: ‡ |
| | | $k \geq 3$ odd | $\star$ | $\Sigma_{k-1}^E$ | $\Pi_k^E$ | H: ‡ , $\in$: t | H/$\in$: ‡ |
| | | $k \geq 2$ even | $\star$ | $\Sigma_k^E$ | $\Pi_{k-1}^E$ | H/$\in$: ‡ | H: ‡, $\in$: t |
| Non-simple | Unique | $k = 1$ | Horn | $\Sigma_1^E$ | ?† | H/$\in$: ‡ | |
| | | | Krom/core | $\Sigma_1^E$ | NL | H: t, $\in$: ‡ | H/$\in$: t |
| | | $k \geq 2$ | $\star$ | $\Sigma_k^E$ | $\Pi_k^E$ | H: t, $\in$: ‡ | |
| | Non-unique | $k \geq 1$ | $\star$ | $\Sigma_k^E$ | $\Pi_k^E$ | H: t, $\in$: [Lohrey, 2012] | |
| $\star$ | $\star$ | $k = \omega$ | $\star$ | AEXP(poly) | AEXP(poly) | H: t, [Hannula et al., 2016], $\in$: [Lohrey, 2012] | |

$\star$: "any", "H": hardness, "$\in$": membership, "t": this paper, †: likely identical with first row. ‡: follows from some other result in the table.

# Conclusion

**Background**: $\mathrm{SAT}, \mathrm{QBF}, \mathrm{DQBF}$ increasingly intractable, while $\mathrm{HORNSAT}, \mathrm{QHORN}, \mathrm{DQHORN}$ all P-complete.

**This paper**: Isolated $\mathrm{DQHORN}$ as a special fragment of $\mathsf{SO}_2$:

(1) *unique* and *simple* function terms
(2) *Horn* clauses
(3) in $\Sigma_1$

Can (1)-(3) be relaxed without increasing complexity?
Answer: No (one exception: replace Horn with Krom/core)

Open question: Complexity of simple unique $\Pi_1$ (dual of $\mathrm{DQHORN}$)?

## Thanks!

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